Pergamon

*Int. J. Multiphase Flow* Vol. 22, No. 6, pp. 1073–1104, 1996

PII: S0301-9322(96)00031-6
0301-9322/96 $15.00 + 0.00

# A FIRST ORDER RELAXATION MODEL FOR THE PREDICTION OF THE LOCAL INTERFACIAL AREA DENSITY IN TWO-PHASE FLOWS

M. MILLIES†, D. A. DREW and R. T. LAHEY, JR

Center for Multiphase Research, Rensselaer Polytechnic Institute, Troy, NY 12180, U.S.A.

*(Received 11 April 1994; in revised form 5 March 1996)*

**Abstract**—Many energy production and chemical processes involve vapor/liquid two-phase flows. Mass and energy are often exchanged between the vapor and the liquid phases, and the fluid mechanics of the two-phase system is strongly influenced by the exchange of momentum between each phase. Significantly, the transport of mass, energy and momentum between the phases takes place across interfaces. Therefore the interfacial area density (i.e. the interfacial area per unit volume) has to be accurately known in order to make reliable predictions of the interfacial transfers. Indeed, the interfacial area density must be known for both steady and transient two-phase flows. It is the purpose of this paper to present a first order relaxation model which is derived from the Boltzmann transport equation, and which accurately describes the evolution of interfacial area density for bubbly flows. In particular, the local, instantaneous interfacial area densities and volume fractions are predicted for vertical flow of a vapor/liquid bubbly flow involving both bubble clusters and individual bubbles. Copyright © 1996 Elsevier Science Ltd.

*Key Words:* interfacial area, coalescence, bubble break-up

## 1. INTRODUCTION

Many researchers have applied various techniques in order to measure the interfacial area density ($A_i'''$) in two-phase flows. Techniques include dynamic gas disengagement (Patel *et al.* 1989; Daly & Patel 1992), noise analysis of transmitted light beams (Mudde *et al.* 1992), neutron radiography (Chang & Harvel 1992), photography (Akita & Yoshida 1974), ultrasound (Bensler *et al.* 1991) and electrical resistivity probe (Lewis & Davidson 1983; Serizawa & Kataoka 1990; Lahey & Lee 1992). The multitude of measurements and techniques reflects the importance of interfacial area density for all multiphase transport processes.

The interfacial area density may be described in a precise way using the delta function and applying time, space or ensemble averaging (Ishii 1975; Ishii & Mishima 1984; Ishii 1992). The ergodic theorem suggests that the specific method of sampling the set of events leads to the same results, provided that a sufficiently large number of independent samples are taken. All methods of averaging lead to an equivalent definition of the interfacial area density. However, a constitutive equation is still required which describes the bubble breakup and coalescence. Two basic approaches have been used herefore, namely, a local approach and a global approach. In the local approach, the bubble size distribution is predicted by modeling the local break-up and coalescence processes of bubbles. Prince & Blanch (1990a, 1990b) described the break-up process based on the probability of liquid eddies colliding with and destroying the individual bubbles. In addition, Marrucci & Nicodemo (1967), Thomas *et al.* (1983) and Prince & Blanch (1990a, 1990b) described the coalescence process based on the probability that a bubble pair stays together long enough to allow for sufficient drainage of the liquid film between the bubbles so that coalescence can occur. Since the coalescence of only two bubbles was considered, these approaches are called binary theories. Prince & Blanch (1990a) and Oolman & Blanch (1986) also investigated the influence of fluid properties on the coalescence rate.

Recently, Stewart *et al.* (1993) presented the bubble size distributions measured by several authors. They found that most bubble size distributions have a pronounced tail, that is, their

†Institut für Verfahrenstechnik, Universität Hannover, Callinstr. 36, D-30167 Hannover, Germany.

distribution is much broader than simple models for bubbly flows would allow. They concluded that bubble clusters, that is, ensembles of bubbles, have a considerable influence, and that coalescence events in these clusters may occur much more frequently than binary theories would predict, resulting in more larger bubbles. Garncarek *et al.* (1991) measured the size of the bubble clusters in two-phase air/water flows. Park (1992) presented photographs of bubble clusters and the bubbly/slug flow regime transition in oil/air flows. These experimental results further stress the importance of bubble clusters in the coalescence process.

Another approach for predicting interfacial area density was to consider the relevant phenomena on a global scale. Ishii & Mishima (1984) presented correlations for the global interfacial area density as a function of void fraction and flow regime. Interestingly, the transport equations presented in this paper become unstable at the transition from bubbly flow to another flow regime. Nagel *et al.* (1972, 1973, 1976, 1978), Calderbank (1967), Burgess & Calerbank (1975), Chandrasekharan & Calderbank (1981) and de Figueiredo & Calerbank (1979) applied Kolmogoroff's theory of turbulence and obtained,

$$\langle A_i''' \rangle \sim \epsilon_{\mathrm{L}}^{0.4} \left(\frac{\sigma}{\rho_{\mathrm{L}}}\right)^{-0.6} \langle \alpha \rangle^m, \qquad [1]$$

where $\langle A_i''' \rangle$ is the global interfacial area density, $\epsilon_{\mathrm{L}}$ is the dissipation rate in the liquid, $\sigma$ is the surface tension, $\rho_{\mathrm{L}}$ is the liquid density, $<\alpha>$ is the cross-sectional averaged void fraction and $m$ is an exponent which can be fitted to experimental results. Thus Nagel *et al.* and Calderbank *et al.* predicted the global interfacial area density as a function of global void fraction and dissipation rate. They have applied [1] to packed beds, pipe reactors, jet tube washers, stirred vessels, pipe flow, bubble columns and dual flow columns. Even though their theoretical approach seemed convincing, the agreement with experimental data was often poor. Nagel *et al.* argued that their approach should have been carried out on a local scale, but they had only global data available. Interestingly, the present work may be thought of as the derivion of an expression similar to [1] on a local scale for the transient process of coalescence and break-up in bubbly two-phase flow, thus connecting the local and the global approaches.

Deckwer (1992) showed that the experimental data for interfacial area density in two-phase flow differ widely. He recommended the correlation by Akita & Yoshida (1974) for the global interfacial area density in bubble columns:

$$\langle A_i''' \rangle = 0.23\left(\frac{\sigma}{\rho_{\mathrm{L}} g}\right)^{-1/2}\left(\frac{g D_{\mathrm{b}}^3 \rho_{\mathrm{L}}^2}{\mu_{\mathrm{L}}^2}\right)^{0.12}\left(\frac{u_{\mathrm{G}}^2}{g D_{\mathrm{b}}}\right)^{0.06}\langle \alpha \rangle. \qquad [2]$$

In this equation, $D_{\mathrm{b}}$ is the diameter of the bubbles, $\mu_{\mathrm{L}}$ is the viscosity of the liquid, and $u_{\mathrm{G}}$ is the velocity of the gas. We will use this equation later on to assess our model.

## 2. TRANSPORT EQUATION FOR THE PROBABILITY DENSITY FUNCTION

Kalkach-Navarro (1992) have proposed a form of the Boltzmann transport equation for the probability density function, $f$, of the bubbles:

$$\frac{\partial f}{\partial t} + \nabla \cdot [f \mathbf{u}_{\mathrm{G}}] = \int_v^\infty b(u) f(u)\,\mathrm{d}u - \int_0^v b(v)\frac{u}{v} f(v)\,\mathrm{d}u + \frac{1}{2}\int_0^v c(u, v-u) f(u) f(v-u)\,\mathrm{d}u - \int_0^\infty c(u,v) f(u) f(v)\,\mathrm{d}u. \qquad [3]$$

Here $f(v)$ denotes the probability density function of $v$, the bubble volume. $f$ is the number density of bubbles per differential bubble volume and thus,

$$N_b''' = \int_0^\infty f(u)\,du, \qquad [4]$$

where $N_b'''$ is the number density of the bubbles, $\mathbf{u}_G$ is the average bubble velocity, and $b$ and $c$ are the breakup and coalescence kernels, respectively. The velocity vector $\mathbf{u}_G$ is obtained from solving the momentum balance for the gas phase simultaneously with [3]. Virtual mass effects, forces leading to turbulent diffusion, lateral forces and wall forces have an influence in the momentum equation, whereas the dependency of the kernels on the flow parameters is discussed next and in more detail in section 3, where bubble clusters are considered further. A similar transport model has also been used by Guido-Lavalle *et al.* (1993, 1994). Fortunately, Prince & Blanch (1990) have developed equations for these coalescence and breakup kernels. Let us recall that coalescence occurs due to contact between bubbles, and the coalescence of bubbles is more probable as (a) the contact time between the bubbles increases and (b) the time needed for the drainage of the liquid film entrapped between the bubbles decreases (e.g. when low viscosity liquids are used). Collisions between bubbles may occur due to (a) the turbulent motion of the liquid, (b) differences in bubble rise velocities and (c) the effect of liquid shear. However, Prince & Blanch (1990) found that the main influence on bubble coalescence was the velocity fluctuations of the liquid. In particular, the higher the liquid phase turbulence level, the more likely it is that the liquid eddies will break up individual bubbles and bubble clusters before the bubbles in these clusters can coalesce. We did not include the effects of turbulent dispersion in [3]. An additional term is obtained in [3] if turbulent dispersion of bubbles is considered.

We used the following simplifying assumptions in order to obtain an analytical solution of the equations given by Prince & Blanch (1990):

1. The collisions of the bubbles occur due to the turbulent fluctuation velocities, the two other mechanisms mentioned above are neglected.
2. The coalescence rate may be described using one equivalent bubble diameter instead of the diameters of both bubbles.
3. The exponential functions in the collision efficiency and the bubble break-up rate are expanded neglecting terms of smaller magnitude at a point of expansion corresponding to the data given by Prince & Blanch (1990).

We obtained,

$$b(u) = b_0, \qquad [5]$$

and

$$c(u, v) = c_0, \qquad [6]$$

where

$$b_0 = b_1 \epsilon_L^{9/5} \left(\frac{\sigma}{\rho_L}\right)^{-11/5} \qquad [7]$$

and

$$c_0 = c_1 \epsilon_L^{-3/5} \left(\frac{\sigma}{\rho_L}\right)^{7/5}. \qquad [8]$$

Here $\epsilon_L$ denotes the local turbulent energy dissipation of the liquid per unit liquid mass and $b_1$ and $c_1$ are coefficients to be determined empirically. Note that the data of Schumpe and Grund (1986) imply $b_1 = 0.0163$ and $c_1 = 0.0369$. Assumptions 1 and 2 would somehow restrict the validity of the equations to flows with a narrow size distribution of the bubbles. But due to the considerations made later on for the bubble clusters, the resulting equations will prove valid also for the general case.

*2.1. Numerical solution of the transport equation*

We will first solve [3] numerically and present some results in order to better understand the transient phenomena involved. We will then solve [3] analytically, obtaining an explicit solution for the steady-state local interfacial area density, $A_i'''$, which we can compare with the results given by Akita & Yoshida (1974).

The numerical evaluation of the Boltzmann transport equation, [3] was carried out using a finite difference method, which separated the bubbles into 100 equally spaced volumes for one-dimensional bubbly two-phase flow, starting from a uniform bubble size. The transient volume distribution was obtained as a result. The volume distribution of the bubbles is shown as a function of the bubble volume and the flow path in figure 1. We see in figure 1 that the initial bubbles coalesce to form larger bubbles having two times the initial volume, then three times and so on, while at the same time the size distribution becomes smoother due to bubble break-up.

*2.2. Analytical solution of the transport equation*

For steady fully-developed conditions, [3] becomes:

$$b_0 \int_v^{\infty} f(u)\,\mathrm{d}u - \frac{v}{2} b_0 f(v) + \frac{1}{2} c_0 \int_0^v f(u) f(v-u)\,\mathrm{d}u - c_0 \int_0^{\infty} f(u)\,\mathrm{d}u f(v) = 0. \qquad [9]$$

It can be verified that the following distribution function satisfies [9]:

$$f(v) = \frac{b_0}{c_0}\,\mathrm{e}^{-\sqrt{(b_0)/(c_0 \alpha_b)}v}. \qquad [10]$$

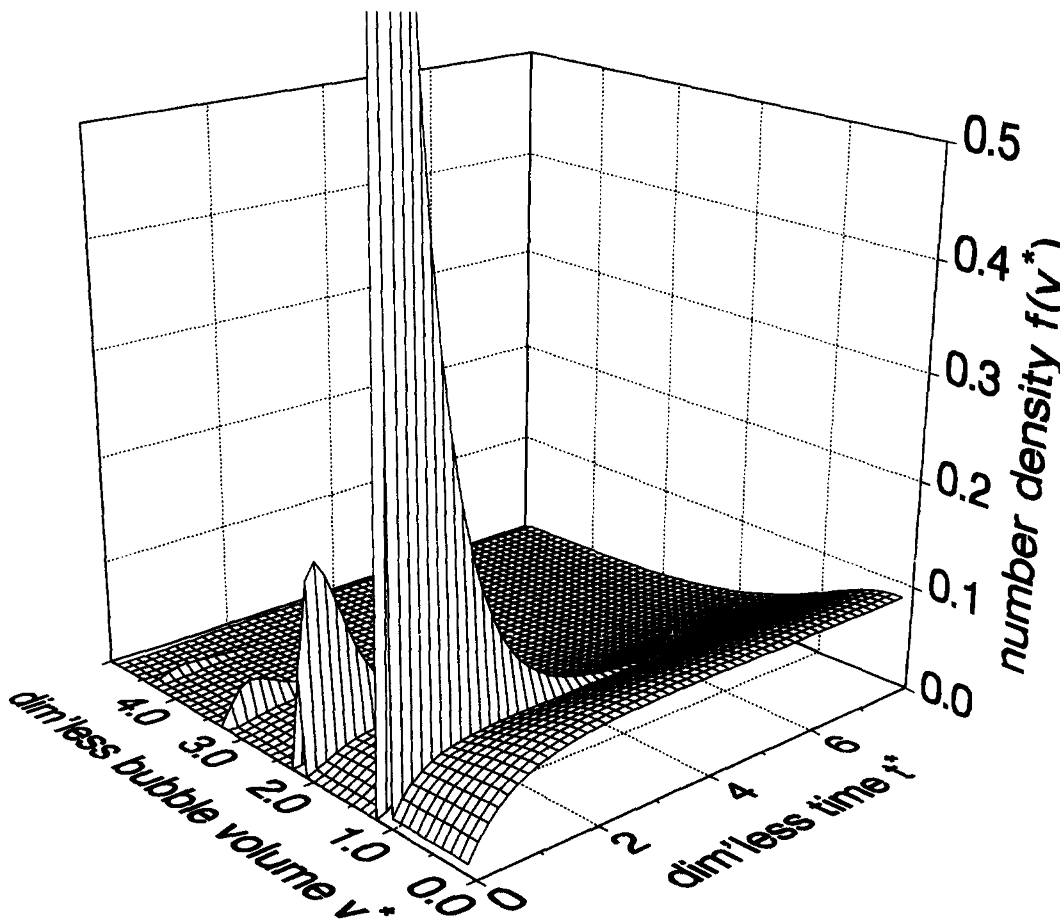


Figure 1. Transient bubble size distribution.

We note that the void fraction of the bubbles is the first moment,

$$\alpha_b = \int_0^\infty u f(u)\,du, \tag{11}$$

which is satisfied by [10].

The interfacial area of a single spherical bubble, having volume $v$, is,

$$a_{ib}(v) = (36\pi)^{1/3} v^{2/3}. \tag{12}$$

Assuming spherical bubbles, the local interfacial area density for a distribution of bubbles becomes:

$$A_{ib}''' = (36\pi)^{1/3} \int_0^\infty u^{2/3} f(u)\,du = \frac{2}{3}(36\pi)^{1/3}\Gamma\left(\frac{2}{3}\right)\left(\frac{b_0}{c_0}\right)^{1/6} \alpha_b^{5/6}, \tag{13}$$

where $\Gamma$ denotes the gamma function, and $\Gamma(2/3) = 1.354$.

Thus, using [7] and [8], [13] becomes

$$A_{ib}''' = 5.41 \epsilon_L^{0.4} \left(\frac{\sigma}{\rho_L}\right)^{-0.6} \alpha_b^{0.83}. \tag{14}$$

We see that the void fraction exponent is different in [14] and [2] for the global and local interfacial area density, respectively. However, the analysis does not account for bubble clusters. This is significant, since when the void fraction increases, bubble clusters become more dominant and coalescence takes place mainly inside the bubble clusters. We will, therefore, underpredict the coalescence rate of high void fractions, by neglecting bubble clusters. This, in turn, will influence the exponent of the void fraction in [14]. Interestingly, Serizawa & Kataoka (1990) gives a void fraction exponent of 0.87 for bubbly two-phase flows in a vertical pipe, which presumably involved both bubbles and bubble clusters.

## 3. MECHANISMS INVOLVING BUBBLE CLUSTERS

This section is concerned with the prediction of bubble cluster size, interfacial area density, bubble cluster formation and destruction rate, etc. All known terms which may cause formation or destruction of the bubble clusters are considered, but only the most important ones are used, so that an analytical solution may be obtained.

The following phenomena may require modeling if accurate predictions are to be made for bubbly flows.

(1) The break-up of bubbles into smaller ones.
(2) The grouping of two bubbles into a size-2 cluster.
(3) The grouping ot $n$ bubbles ($n > 2$) into a size-$n$ cluster.
(4) The coalescence of two single bubbles.
(5) The coalescence of two bubbles inside a cluster.
(6) The coalescence of more than two bubbles inside a cluster.
(7) The simultaneous coalescence of all bubbles inside a cluster.
(8) The removal of one bubble from a cluster.
(9) The removal of several bubbles from a cluster.
(10) The complete break-up of a bubble cluster.
(11) The break-up of a bubble inside a cluster, leaving a cluster containing more bubbles.
(12) The break-up of a bubble inside a cluster, leaving scattered single bubbles.
(13) The uptake of a bubble by a cluster.
(14) The uptake of another cluster by a cluster.

Bubble breap-up events need to be considered in order to balance the coalescence events. Mechanisms (11) and (12) are not likely, so we consider only mechanism (1) for bubble break-up.

The coalescence of two bubbles happens when two bubbles form a size-2 cluster, and subsequently undergo liquid film drainage and interface rupture. Mechanism (4), direct coalescence of two bubbles, can be considered to be composed of the formation of a size-2 cluster, followed immediately by the rupture of the interface. Furthermore, mechanism (3), the simultaneous clustering of more than two bubbles, can be thought of as the repeated uptake of individual bubbles by a cluster. We therefore consider mechanisms (2) and (13), but not mechanism (3). The coalescence of several bubbles, or of all bubbles, in a cluster may be thought as successive coalescence of various bubble-pairs in a cluster. Thus we may neglect mechanisms (6) and (7). Experimental observations (Kalkach-Navarro 1992) show that several bubbles may be removed when an energetic liquid eddy impacts a cluster. Indeed, such eddies may destroy small bubble clusters completely, or make large clusters smaller. Nevertheless, we may describe this as several removals of a single bubble from a cluster, and thus we may consider mechanism (8) while neglecting mechanisms (9) amd (10). The uptake of a bubble cluster by another cluster is considered to be much less likely than the uptake of single bubbles, therefore we also neglect mechanism (14). Thus, we shall consider only five mechanisms: (1), (2), (5), (8) and (13).

### *3.1. The break-up of single bubbles*

Prince & Blanch (1990b) describe the break-up process of a bubble in two steps: he first modeled the probability for the impact of a liquid eddy with a bubble, and then modeled the probability that this impact will cause a break-up. We may write this process symbolically as

$$R_1 : b \rightarrow b + b. \qquad [15]$$

The break-up process of single bubbles has nothing to do with clusters, so we can apply the result from the previous section and find that the bubble cluster breakup rate is:

$$R_1 = r_1 \bar{v}_b N_b''', \qquad [16]$$

where $\bar{v}_b$ is the average volume of the bubbles outside of the clusters, and the coefficient $r_1$ is just the break-up kernel in [3], thus

$$r_1 = b_1 \epsilon_L^{9/5} \left( \frac{\sigma}{\rho_L} \right)^{-11/5}, \qquad [17]$$

Equation [16] will hold only if the bubbles are not strongly restricted by walls, such as in the case of a Taylor-bubble, so that the break-up frequency does not increase with bubble volume.

### *3.2. The grouping of two bubbles into a size-2 cluster*

We may write the bubble cluster mechanism as

$$R_2 : b + b \rightarrow Cl[2] \qquad [18]$$

As discussed previously, Prince & Blanch (1990b) proposed three mechanisms which can bring bubbles together so that a cluster may be formed: (a) turbulent liquid velocity fluctuations, (b) the different relative velocities of the bubbles due to different bubble sizes, and (c) the different velocities of the bubbles at neighboring positions due to a shear field in the liquid. Prince showed that the first mechanism dominates bubble collisions. Nevertheless, the second mechanism will be important for the uptake of bubbles into clusters, since the relative velocities of bubbles and clusters are quite different, and the wake of the cluster can be quite large. For low void fractions and low dissipation rates, the rate of bubble grouping events is the same as the coalescence rate. We see this as follows: since the dissipation rate is low, there are few liquid eddies, and the bubble clusters are not knocked apart. Since the void fraction is small, it is not very probable that the bubble pairs (i.e. clusters) may take up further bubbles, thus for all bubble pairs, there is only one further thing that can happen; they all coalesce after a sufficient time. We therefore may use our previous findings for the coalescence rate to describe the grouping of bubbles into bubble pairs as:

$$R_2 = r_2 N_b'''^2, \qquad [19]$$

where the coefficient $r_2$ is half of the coalescence kernel in [3]

$$r_2 = \tfrac{1}{2} c_1 \epsilon_L^{-3/5} \left(\frac{\sigma}{\rho_L}\right)^{7/5}. \qquad [20]$$

The factor $\frac{1}{2}$ in [20] avoids double counting of grouping events. Equation [19] implies the assumption that the grouping mechanism is statistical, and therefore proportional to the probability of a binary interaction, which is proportional to the number of density squared. There are some conditions when bubbles collide less frequently then predicted by [19]. This includes situations of flow in very narrow tubes at low gas loads, or at very low superficial gas velocities in the so-called homogeneous flow regime in a bubble column where very small bubbles without considerable coalescence may be obtained under very retricted conditions.

### 3.3. *The coalescence of two bubbles inside a cluster*

The mechanism is given symbolically by

$$R_5: Cl[n] \rightarrow Cl[n-1]. \qquad [21]$$

We note, that for very small void fraction, mechanisms (2) and (5) reduce to the coalescence mechanism for single bubbles

$$R_5: Cl[2] \rightarrow b. \qquad [22]$$

The coalescence rate in size-2 clusters is obtained from the number per unit volume of size-2 clusters, $Cl[2]$, divided by the average time needed for interstitial liquid film drainage. Prince & Blanch (1990b) give an expression for the average liquid film drainage time:

$$\tau_d = \frac{1}{d_l} \bar{v}_b^{1/2} \left(\frac{\sigma}{\rho_L}\right)^{-1/2}, \qquad [23]$$

where Prince gives $d_l = 1.18$. The coalescence rate is then given by

$$R_5 = d_l \bar{v}_b^{-1/2} \left(\frac{\sigma}{\rho_L}\right)^{-1/2} Cl[2] = r_5 Cl[2], \qquad [24]$$

where

$$r_5 = d_l \bar{v}_b^{-1/2} \left(\frac{\sigma}{\rho_L}\right)^{-1/2}. \qquad [25]$$

Interestingly, [23] does not include the influence of liquid viscosity. In a separate study Kalkach-Navarro (1992) applied the model given by Thomas *et al.* (1983) which gives the mean liquid film drainage time as

$$\tau_d = \frac{3}{32\pi} \mu_L \rho_L \epsilon_L^{2/3} \frac{D_b^{14/3}}{(\sigma h_c)^2}, \qquad [26]$$

where $D_b$ is the bubble diameter, $h_c$ is the critical film thickness and $\mu_L$ is the viscosity of the liquid phase. We did not apply this equation, since the critical film thickness, $h_c$, is unknown and will depend itself on fluid properties. Nevertheless, we obtained good agreement with experimental data applying [24].

For the general case, describing the coalescence of a bubble pair within a size-$n$ cluster

$$R_5: Cl[n] \rightarrow Cl[n-1], \qquad [27]$$

we assume that the number of coalescence events increases linearly with the number of bubbles in the cluster, thus

$$R_5 = r_5 (n-1) Cl[n]. \qquad [28]$$

Significantly, detailed comparisons with data, to be discussed later, indicated the best prediction of bubble cluster sizes occurred when we used:

$$d_1 = 0.429. \tag{29}$$

Hence, [29] is the recommended parameter to be used in [25] and [28].

*3.4. The removal of one bubble from a cluster*

This mechanism is given symbolically by

$$R_8 = Cl[n] \rightarrow Cl[n-1] + b. \tag{30}$$

Levich (1962) has modeled the length of time that two bubbles remain together before being knocked apart by the liquid phase turbulence as:

$$\tau_b = \frac{1}{d_2} \bar{v}_b^{2/9} \epsilon_L^{-1/3}, \tag{31}$$

where $d_2$ is a constant. This equation has been applied by Prince & Blanch (1990) and Kalkach-Navarro *et al.* (1992, 1993). We may thus describe the removal of one bubble out of a size-2 cluster as:

$$R_8 = Cl[2] \frac{1}{\tau_b}. \tag{32}$$

We assume that the rate of removal events increases with the number of bubbles inside a cluster, since the probability of a liquid eddy colliding with one of the bubbles in the cluster increases. Assuming a linear dependence, and noting that the minimum size cluster is $n = 2$, we obtain:

$$R_8 = Cl[n] \frac{1}{\tau_b}(n-1) = d_2 \epsilon_L^{1/3} \bar{v}_b^{-2/9} Cl[n](n-1) = r_8(n-1)Cl[n]. \tag{33}$$

Fitting the constant $d_2$ from experimental data (1986) we obtain:

$$d_2 = 4.94 \times 10^{-9}. \tag{34}$$

*3.5. The uptake of a bubble by a cluster*

This process is represented by:

$$R_{13}: Cl[n] + b \rightarrow Cl[n+1]. \tag{35}$$

Prince & Blanch (1990b) found that the grouping of two bubbles to form a size-2 cluster is mainly due to turbulent liquid fluctuation velocities, since the differences in the bubble rise velocities are relatively small. We have already considered this in mechanism (2), where the dissipation rate in the liquid associated with the turbulent fluctuation velocities was used for modeling. In this case, the rise velocities of the bubble clusters and the individual bubbles are quite different; indeed, clusters overtake bubbles and may then capture them in their wake (Kalkach-Navarro, 1992). We therefore use the second of the mechanisms considered by Prince & Blanch (1990b):

$$R_{13} = N_b''' n Cl[n] \bar{v}_b^{2/3} |\bar{u}_{R,Cl} - \bar{u}_{R,b}|, \tag{36}$$

where the relative velocity of vapor field-$k$ with respect to the liquid is given by:

$$\bar{u}_{R,k} = \bar{u}_k - \bar{u}_L. \tag{37}$$

Due to a lack of better information, we have assumed herein that the increase of the relative velocity of the clusters, and the consequent increase of the size of the wake of the clusters, which increases the rate of bubble capture, is directly proportional to the number of bubbles inside the cluster times the relative velocity difference between the clusters ($\bar{u}_{Cl}$) and single bubbles ($\bar{u}_b$). Thus we obtain,

$$R_{13} = r_{13} N_b''' n Cl[n], \tag{38}$$

where

$$r_{13} = |\bar{u}_{R,Cl} - \bar{u}_{R,b}| \bar{v}_b^{2/3}. \tag{39}$$

## 4. THE STEADY STATE SOLUTION

### 4.1. *Number density of clusters*

The number of clusters containing between 2 and $n$ bubbles may change because:

- single bubbles may group to form size-2 clusters,
- the bubbles in size-2 or size-$(n + 1)$ clusters can coalesce,
- one bubble may be removed from a size-2 or size-$(n + 1)$ cluster, or
- one bubble may be taken up into a size-$n$ cluster.

Thus, using [19], [28], [33] and [38] we obtain the following balance law for bubble clusters:

$$\underbrace{r_2 N_b'''^2}_{\text{Grouping}} - \underbrace{r_5 Cl[2]}_{\text{Coalescence}} - \underbrace{r_8 Cl[2]}_{\text{Removal}}$$

$$+\underbrace{r_8 Cl[n+1]n}_{\text{Removal}} + \underbrace{r_5 Cl[n+1]n}_{\text{Coalescence}} - \underbrace{r_{13} N_b''' Cl[n]n}_{\text{Uptake}} = 0. \qquad [40]$$

The total number of bubbles is assumed to be bounded. This implies that $nCl[n] \to 0$ as $n \to \infty$. The right hand side of [40] must become zero as $n \to \infty$. Therefore

$$Cl[2] = \frac{r_2 N_b'''^2}{r_5 + r_8}. \qquad [41]$$

Also, from [40]

$$Cl[n+1] = Cl[n]\frac{r_{13} N_b'''}{r_5 + r_8}. \qquad [42]$$

Solving [42], with the condition [40] for $Cl[2]$, gives

$$Cl[n] = \frac{r_2}{r_{13}} N_b''' \chi^{n-1}, \qquad [43]$$

where

$$\chi = \frac{r_{13} N_b'''}{r_5 + r_8} = \frac{r_{13}\alpha_b}{\bar{v}_b(r_5 + r_8)}. \qquad [44]$$

The total number density of bubble clusters is thus:

$$\sum_{i=2}^{\infty} Cl[i] = \frac{r_2}{r_{13}} N_b''' \frac{\chi}{1-\chi} \qquad [45]$$

and the total number of bubbles in these bubble clusters per unit volume is

$$\sum_{i=2}^{\infty} iCl[i] = \frac{r_2}{r_{13}} N_b''' \frac{\partial}{\partial \chi} \sum_{i=2}^{\infty} \chi^i = \frac{r_2}{r_{13}} N_b''' \frac{\chi(2-\chi)}{(1-\chi)^2}. \qquad [46]$$

### 4.2. *The average volume of bubbles in the bubble clusters*

The total gas volume in bubble clusters containing $n$ or more bubbles increases due to the uptake of bubbles into the clusters and decreases due to removal of bubbles from the clusters, but coalescence affects the total gas volume only if two bubbles in a size-$n$ cluster coalesce, since after coalescence this cluster will be of size $n - 1$ and thus will no longer contribute to the gas volume in bubble clusters having $n$ or more bubbles. For $n = 2$, using [19], [28], [33] and [38] we have the following balance law for gas volume:

$$\underbrace{r_2 N_b'''^2 2\bar{v}_b}_{\text{Grouping}} - \underbrace{r_5 Cl[2]2\bar{v}_{Cl}[2]}_{\text{Coalescence}} - \underbrace{r_8 Cl[2]2\bar{v}_{Cl}[2]}_{\text{Removal}} =$$

$$\underbrace{-r_{13} N_b''' \sum_{i=2}^{\infty} iCl[i]\bar{v}_b}_{\text{Uptake}} + \underbrace{r_8 \sum_{i=3}^{\infty} (i-1)Cl[i]\bar{v}_{Cl}[i]}_{\text{Removal}}, \qquad [47]$$

similarly, for $n > 2$:

$$\underbrace{r_{13} N_b''' Cl[n-1](n-1)((n-1)\bar{v}_{Cl}[n-1] + \bar{v}_b)}_{\text{Uptake from } Cl[n-1]}$$

$$\underbrace{-r_5 Cl[n](n-1)n\bar{v}_{Cl}[n]}_{\text{Coalescence}} - \underbrace{r_8 Cl[n](n-1)n\bar{v}_{Cl}[n]}_{\text{Removal}}$$

$$= \underbrace{-r_{13} N_b''' \sum_{i=n}^{\infty} iCl[i]\bar{v}_b}_{\text{Uptake}} + \underbrace{r_8 \sum_{i=n+1}^{\infty} (i-1)Cl[i]\bar{v}_{Cl}[i]}_{\text{Removal}}. \qquad [48]$$

Assuming that the average bubble volume is the same in all size bubble clusters

$$\bar{v}_{Cl}[i] = \bar{v}_{Cl}[j] \equiv \bar{v}_{Cl}; \quad i, j > 2. \qquad [49]$$

And using [42]

$$r_{13} N_b''' Cl[n-1] = Cl[n](r_8 + r_5), \qquad [50]$$

we find that

$$Cl[n](r_5 + r_8)(n-1)(\bar{v}_b - \bar{v}_{Cl}) = \sum_{i=n+1}^{\infty} (i-1)Cl[i](r_8\bar{v}_{Cl} - (r_5 + r_8)\bar{v}_b), \qquad [51]$$

where $\bar{v}_{Cl}$ is the average bubble volume inside the clusters. We may evaluate the summation using [43]:

$$\sum_{i=n+1}^{\infty} (i-1)Cl[i] = \frac{r_2}{r_{13}} N_b''' \sum_{i=n+1}^{\infty} (i-1)\chi^{i-1} = \frac{r_2}{r_{13}} N_b''' \left( \frac{\partial}{\partial \chi} \sum_{i=n+1}^{\infty} \chi^i - \sum_{i=n+1}^{\infty} \chi^{i-1} \right)$$

$$= \frac{r_2}{r_{13}} N_b''' \left[ \frac{\partial}{\partial \chi} \left( \frac{\chi^{n+1}}{1-\chi} \right) - \frac{\chi^n}{1-\chi} \right]$$

$$= \frac{r_2}{r_{13}} N_b''' \frac{(n+1)(1-\chi) + \chi - (1-\chi)}{(1-\chi)^2} \chi^n$$

$$= \frac{r_2}{r_{13}} N_b''' \chi^n \left( \frac{n-1}{1-\chi} + \frac{1}{(1-\chi)^2} \right). \qquad [52]$$

Neglecting the second term in the parentheses in [52], which is small, we obtain from [51]:

$$\frac{r_2}{r_{13}} N_b''' \chi^{n-1}(r_5 + r_8)(n-1)(\bar{v}_b - \bar{v}_{Cl}) = \frac{r_2}{r_{13}} N_b''' \chi^n \frac{n-1}{1-\chi} (r_8\bar{v}_{Cl} - (r_5 + r_8)\bar{v}_b). \qquad [53]$$

Hence, we obtain for the average volume of bubbles inside the clusters:

$$\bar{v}_{Cl} = \frac{(r_5 + r_8)\bar{v}_b\left(1 + \dfrac{\chi}{1-\chi}\right)}{r_8\dfrac{\chi}{1-\chi} + r_5 + r_8} \doteq \bar{v}_b \frac{1}{\left(1 - \dfrac{r_5}{r_5 + r_8}\chi\right)}. \quad [54]$$

For the average bubble volume for size-2 clusters, using [41] and [48], we find:

$$r_2 N_b'''^2(2\bar{v}_b - 2\bar{v}_{Cl}[2]) = \sum_{i=3}^{\infty}(i-1)Cl[i](r_8\bar{v}_{Cl} - (r_5 + r_8)\bar{v}_b), \quad [55]$$

or from [52], using [44]:

$$2r_2 N_b'''^2(\bar{v}_b - \bar{v}_{Cl}[2]) = \frac{r_2 N_b'''^2}{r_5 + r_8}\chi\frac{2-\chi}{(1-\chi)^2}(r_8\bar{v}_{Cl} - (r_5 + r_8)\bar{v}_b). \quad [56]$$

Thus the average volume of bubbles in size-2 clusters using [54] is:

$$\bar{v}_{Cl}[2] = \bar{v}_b \frac{1 + \dfrac{\chi^2}{2(1-\chi)^2}}{1 - \dfrac{r_5}{r_5 + r_8}\chi}, \quad [57]$$

which is the same as the average bubble volume in the other bubble clusters if we neglect the second term in the numerator. Thus,

$$\bar{v}_{Cl}[2] = \bar{v}_{Cl}. \quad [58]$$

We may now evaluate the void fraction of the clusters:

$$\alpha_{Cl} = \sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}. \quad [59]$$

We use [52] and [50] to obtain:

$$\alpha_{Cl} = \frac{r_2}{r_{13}} N_b''' \frac{\chi(2-\chi)}{(1-\chi)^2}\bar{v}_b \frac{1}{1 - \dfrac{r_5}{r_5 + r_8}\chi}. \quad [60]$$

Since the number of bubbles outside of clusters times their average volume is the void fraction,

$$N_b'''\bar{v}_b = \alpha_b, \quad [61]$$

we obtain for the volume fraction of the gas in the bubble clusters:

$$\alpha_{Cl} = \alpha_b \frac{r_2}{r_{13}} \frac{\chi(2-\chi)}{(1-\chi)^2\left(1 - \dfrac{r_5}{r_5 + r_8}\chi\right)}. \quad [62]$$

We may now evaluate the average number of bubbles per cluster:

$$\bar{n} = \frac{\sum_{i=2}^{\infty} iCl[i]}{\sum_{i=2}^{\infty} Cl[i]} = \frac{\dfrac{r_2}{r_{13}} N_b''' \dfrac{\chi(2-\chi)}{(1-\chi)^2}}{\dfrac{r_2}{r_{13}} N_b''' \dfrac{\chi}{1-\chi}} = \frac{2-\chi}{1-\chi} = 1 + \frac{1}{1-\chi}. \quad [63]$$

Let us next consider the average relative velocity of clusters:

$$\bar{u}_{R,Cl} = \frac{\sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}\bar{u}_{R,Cl}[i]}{\sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}}. \quad [64]$$

We note in [64] that the average bubble volume, $\bar{v}_{Cl}$, will cancel out, since it is the same for all clusters. We assume that, due to buoyancy, the relative velocity of a cluster is proportional to the gas volume in the cluster, thus

$$\bar{u}_{R,Cl}[i] = \bar{u}_{R,b} i\bar{v}_{Cl}/\bar{v}_b. \quad [65]$$

Hence from [54],

$$\bar{u}_{R,Cl} \equiv \bar{u}_{Cl} - \bar{u}_L = \frac{\frac{r_2}{r_{13}} N_b''' \sum_{i=2}^{\infty} i\chi^{i-1} i \frac{1}{1 - \frac{r_5}{r_5 + r_8}\chi} \bar{u}_{r,b}}{\frac{r_2}{r_{13}} N_b''' \sum_{i=2}^{\infty} i\chi^{i-1}}$$

$$= \frac{\bar{u}_{r,b}}{1 - \frac{r_5}{r_5 + r_8}\chi} \frac{\chi \frac{\partial^2}{\partial \chi^2} \sum_{i=0}^{\infty} \chi^i + \frac{\partial}{\partial \chi} \sum_{i=2}^{\infty} \chi^i}{\frac{\partial}{\partial \chi} \sum_{i=2}^{\infty} \chi^i}$$

$$= \frac{\bar{u}_{r,b}}{1 - \frac{r_5}{r_5 + r_8}\chi} \left(1 + \frac{(1-\chi)^2}{\chi(2-\chi)} \chi \frac{2}{(1-\chi)^3}\right)$$

$$= \frac{\bar{u}_{r,b}}{1 - \frac{r_5}{r_5 + r_8}\chi} \left(1 + \frac{2}{(2-\chi)(1-\chi)}\right). \quad [66]$$

In subsequent developments we will assume for simplicity that the average velocity of the bubble clusters, $\bar{u}_{Cl}$, is the same for the transport of bubble cluster volume fraction ($\alpha_{Cl}$) and for the number density of clusters ($\Sigma\ Cl$).

### 4.3. *Interfacial area density*

The geometrical interfacial area density is defined by:

$$A'''_{iCl} = \bar{a}_{ib} \sum_{i=2}^{\infty} Cl[i] i (\bar{v}_{Cl}/\bar{v}_b)^{2/3}, \quad [67]$$

where $a_{ib}$ is the average interfacial area of a single bubble outside the cluster, averaged over the size distribution of the bubbles. Actually, the interfacial area density of some of the bubbles in the cluster will not be available for momentum, mass and heat transfer, since they are shielded from the continuous phase by the other bubbles. These self-shielding effects cause the effective interfacial area density of the clusters which is responsible for momentum, heat and mass transfer to differ from the geometrical interfacial area density. Therefore, we define the effective interfacial area density of the clusters as:

$$A'''_{iCl} = \bar{a}_{ib} \sum_{i=2}^{\infty} Cl[i] i^{m} (\bar{v}_{Cl}/\bar{v}_b)^{2/3}, \quad [68]$$

where $m = 1$ implies that each bubble contributes equally to the interfacial area density; which is the geometric interfacial area density. The exponent $m$ must be smaller than unity, however the value of the correct exponent is not known, since it is not easy to measure the interfacial area density which is actually available for momentum, heat and mass transfer. Indeed this requires the use of complicated chemical measurement techniques. As a result, the geometric interfacial area density was used for this study. Using [46] we obtain:

$$A'''_{iCl} = a_{ib}\frac{r_2}{r_{13}}N'''_b\frac{\chi(2-\chi)}{(1-\chi)^2}\left(\frac{1}{1-\dfrac{r_5}{r_5+r_8}\chi}\right)^{2/3}. \qquad [69]$$

Note that

$$a_{ib}N'''_b = A'''_{ib}. \qquad [70]$$

Therefore,

$$A'''_{iCl} = A'''_{ib}\frac{r_2}{r_{13}}\frac{\chi(2-\chi)}{(1-\chi)^2}\left(1-\frac{r_5}{r_5+r_8}\chi\right)^{-2/3}. \qquad [71]$$

*4.4. The average bubble volume*

We have already discussed the balance law for the gas volume which is taken up into all the clusters and removed from them. A mass balance for the bubbles outside of the clusters gives no additional information, but a balance for the number of these bubbles does. The number of bubbles outside the clusters is increased by the break-up of bubbles, by the removal of bubbles out of clusters and by coalescence in size-2 clusters. It is decreased due to the uptake of bubbles into clusters and the grouping of bubbles into size-2 clusters. The appropriate balance law is:

$$\underbrace{\tfrac{1}{2}r_1\bar{v}_bN_b}_{\text{Break-up}} + \underbrace{r_8\sum_{i=3}^{\infty}(i-1)Cl[i] + r_82Cl[2]}_{\text{Removal}} + \underbrace{r_5Cl[2]}_{\text{Coalescence}}$$

$$= \underbrace{2r_2N'''^2_b}_{\text{Grouping}} + \underbrace{r_{13}N'''_b\sum_{i=2}^{\infty}iCl[i]}_{\text{Uptake}}, \qquad [72]$$

where the first term will be discussed in the next section. Using [44], [46], [50] and [41] we obtain:

$$\frac{\bar{v}_b}{N'''_b} = 4\frac{r_2}{r_1} + 2\frac{r_{13}}{r_1}\frac{r_2}{r_{13}}\frac{\chi(2-\chi)}{(1-\chi)^2} - 2\frac{r_{13}}{r_1}\frac{r_8}{r_5+r_8}\frac{r_2}{r_{13}}\frac{\chi(2-\chi)}{(1-\chi)^2} - 4\frac{r_2}{r_1}\frac{r_8}{r_5+r_8} - 2\frac{r_2}{r_1}\frac{r_5}{r_5+r_8}$$

$$= 2\frac{r_2r_5}{r_1(r_5+r_8)}\left(1+\frac{\chi(2-\chi)}{(1-\chi)^2}\right) = \frac{2}{(1-\chi)^2}\frac{r_2r_5}{r_1(r_5+r_8)}. \qquad [73]$$

Finally [61] yields:

$$\bar{v}_b = \frac{1}{1-\chi}\sqrt{2\frac{r_2r_5}{r_1(r_5+r_8)}\alpha_b}. \qquad [74]$$

We may iteratively calculate the average bubble volume, $\bar{v}_b$, as follows:

(1) Makes a guess for $\bar{v}_b$.

(2) Evaluate the factors, $r_1$, $r_2$, $r_5$, $r_8$ and $r_{13}$, from [17], [20], [29], [34] and [39], respectively.

Table 1. Summary of the equations to predict the steady-state interfacial area densities

| | | |
|---|---|---|
| | **Coefficients for the different mechanisms** | |
| Break-up: | $r_1 = 0.0163\epsilon_L^{9/5}\left(\frac{\sigma}{\rho_L}\right)^{-11/5}$ | [17] |
| Grouping: | $r_2 = \frac{1}{2}0.0739\epsilon_L^{-3/5}\left(\frac{\sigma}{\rho_L}\right)^{7/5}$ | [20] |
| Coalescence: | $r_5 = 0.429\bar{v}_b^{-1/2}\left(\frac{\sigma}{\rho_L}\right)^{1/2}$ | [29] |
| Removal: | $r_8 = 4.94\ 10^{-9}\epsilon_L^{1/3}\bar{v}_b^{-2/9}$ | [34] |
| Uptake: | $r_{13} = \bar{\mathbf{u}}_{R,Cl} - \bar{\mathbf{u}}_{R,b}\vert\bar{v}_b^{2/3}$ | [39] |
| | **The interfacial area densities** | |
| Bubbles: | $A'''_{ib} = 5.41\epsilon_L^{0.4}\left(\frac{\sigma}{\rho_L}\right)^{-0.6}\alpha_b^{0.83}$ | [13] |
| Clusters: | $A'''_{iCl} = A'''_{ib}G(\chi)$ | [71] |
| where, | $G(\chi) = \frac{r_2\chi(2-\chi)}{r_{13}(1-\chi)}\left(1 - \frac{r_5}{(r_5+r_8)}\chi\right)^{-3/2}$ | |
| and | $\chi = \frac{r_{13}N'''_b}{(r_5+r_8)}$ | [44] |
| | **The average bubble volume** | |
| Bubbles: | $\bar{v}_b = \frac{1}{1-\chi}\sqrt{2\frac{r_2r_5}{r_1(r_5+r_8)}\alpha_b}$ | [74] |
| Clusters: | $\bar{v}_{Cl} = \bar{v}_b\frac{1}{1-\frac{r_5}{r_5+r_8}\chi}$ | [54] |
| | **Volume fractions** | |
| Bubbles: | $\alpha_b = \bar{v}_bN'''_b$ | [61] |
| Clusters: | $\alpha_{Cl} = \alpha_b\frac{r_2}{r_{13}}\frac{\chi(2-\chi)}{(1-\chi)^2\left(1-\frac{r_5}{r_5+r_8}\chi\right)}$ | [62] |

(3) Make a guess for $\chi$, $(0 < \chi < 1)$.
(4) Evaluate the bubble void fraction, $\alpha_b$, from [44].
(5) Evaluate the volume fraction of the clusters, $\alpha_{Cl}$, from [60].
(6) Iterate from step (3) until the total void fraction, $\alpha$, is $\alpha = \alpha_b + \alpha_{Cl}$.
(7) Evaluate $\bar{v}_b$ from [74].
(8) Iterate on $\bar{v}_b$ starting from step (1) until the process converges.

The equations, which are required to evaluate the steady state solution, are summarized in table 1.

### 4.5. *Size distribution*

Let us now reconsider the size distribution implied by our model and relate it to previous work done by Kalkach-Navarro (1992). We note that terms of the form:

$$e^{-v/\bar{v}_b(1 - r_5/(r_5+r_8)\chi)} = e^{-v/\bar{v}_b}\, e^{v/\bar{v}_b(r_5/(r_5+r_8))\chi} \quad [75]$$

occur. If we make a Taylor expansion of the second exponential function on the right hand side of [75] and consider only the terms up to first order we have:

$$e^{-v/\bar{v}_b(1 + r_5/(r_5+r_8)\chi)} \approx e^{-(v/\bar{v}_b)} + \frac{r_5}{r_5+r_8}\chi\frac{v}{\bar{v}_b}e^{-(v/\bar{v}_b)}. \quad [76]$$

We may then obtain an analytical solution for the size distributions under this approximation. Let us rewrite the Boltzmann transport equation [3], given by Kalkach-Navarro (1992):

$$\frac{\partial f}{\partial t} + \nabla \cdot [\mathbf{u}_G f] = \int_v^{\infty} r_1 f(u)\,\mathrm{d}u - r_1 \frac{v}{2} f(v) + \frac{1}{2}\int_0^v cf(u)f(v-u)\,\mathrm{d}u - \int_0^{\infty} cf(u)\,\mathrm{d}uf(v), \qquad [77]$$

where we have substituted $r_1$ for $b(u)$ to describe the bubble break-up mechanism. This equation gives bubble break-up and coalescence for small void fractions if we identify $f$ with the number density of bubbles outside the clusters. We note that Kalkach-Navarro (1992) used $f$ as the overall number density for both the bubbles and the clusters. Actually, as observed by Park (1992), the coalescence mechanism consists of two separate steps: formation of size-2 clusters from single bubbles and then coalescence inside of the clusters. We therefore use the grouping of two bubbles, mechanism (2), instead of the second coalescence term in [77]:

$$-\int_0^{\infty} cf(u)\,\mathrm{d}uf(v) = -2r_2 N_b''' f(v). \qquad [78]$$

We obtain instead of the first coalescence term in [77] a source term for bubble coalescence in size-2 clusters:

$$\frac{1}{2}\int_0^v cf(u)f(v-u)\,\mathrm{d}u = r_5 Cl[2]\int_0^v f[2](u)f[2](v-u)\,\mathrm{d}u, \qquad [79]$$

where $f[2]$ is the normalized size distribution of bubbles in clusters containing two bubbles. We remark that using [41] this term may be brought into the form given by Kalkach-Navarro, [77], for the limiting case of nearly no clusters. We obtain one more term due to removal of bubbles from size-2 clusters:

$$2r_8 Cl[2]f[2]. \qquad [80]$$

The transport equation for bubbles outside of the clusters becomes, if we also assume that the up-take of bubbles into the clusters and removal of bubbles out of the clusters occurs with all clusters:

$$\frac{\partial f}{\partial t} + \nabla \cdot [\mathbf{u}_G f] = \int_0^{\infty} r_1 f(u)\,\mathrm{d}u - r_1 \frac{v}{2} f(v) - 2r_2 \int_0^{\infty} f(u)\,\mathrm{d}uf(v)$$

$$- r_{13}\sum_{i=2}^{\infty} iCl[i]f(v) + r_5 Cl[2]\int_0^v f[2](u)f[2](v-u)\,\mathrm{d}u$$

$$+ 2r_8 Cl[2]f[2] + r_8 \sum_{i=3}^{\infty}(i-1)Cl[i]f[i]. \qquad [81]$$

The stationary solution for the bubbles is to a good first order approximation:

$$f(v) = \frac{N_b'''}{\bar{v}_b}\,\mathrm{e}^{-v/\bar{v}_b} \qquad [82]$$

and, for the bubble clusters

$$f_{Cl}(v) = \frac{1}{\bar{v}_{Cl}}\,\mathrm{e}^{-v/\bar{v}_{Cl}}. \qquad [83]$$

Introducing [82] and [83] in [81] and using [76], yields two linear dependent equations in $\mathrm{e}^{-v/\bar{v}_b}$ and $(v/\bar{v}_b)\mathrm{e}^{-v/\bar{v}_b}$. Except for the information lost due to the approximations used in this section, the resulting equations for the average bubble volume, $\bar{v}_b$, are identical with [74]. The same procedure may be done with the transport equations for the clusters, resulting in equations for the average bubble volume inside the clusters, $\bar{v}_{Cl}$, identical to [54]. That is, we used kernel functions which were independent of the actual bubble diameter, and, as may be justified from experimental results published by Prince & Blanch (1990), only dependent on the average bubble diameter. This results in exponential functions for the size distributions, and calculations with the average bubble volume in the balance equations for the gas volume gives the same result as a calculation done with size distributions. A discussion of how to evaluate the balance equations if the kernels are functions of bubble diameter may be found in Kalkach-Navarro (1992).

### 4.6. Comparison with experimental results

Schumpe and Grund (1986) measured the void fraction of bubble clusters and individual bubbles in a bubble column having a 30 cm inner diameter. The column was 4.4 m high, so that entrance effects may be neglected except for the case of very small superficial gas velocities. The total void fraction and the superficial gas velocity were used as an input for the calculation. The dissipation rate was obtained from Prince *et al.* (1990b) as

$$\epsilon_L = j_{G0} g, \qquad [84]$$

where $j_{G0}$ is the superficial gas velocity and $g$ is gravity.

The measured and predicted values of the void fractions are presented in figure 2. The agreement between the measured and predicted values of the total void fraction is expected, since the total void fraction was used as an input for the calculation. However, the predicted bubble and bubble cluster void fraction is also quite good, which supports our modeling assumptions. Even though

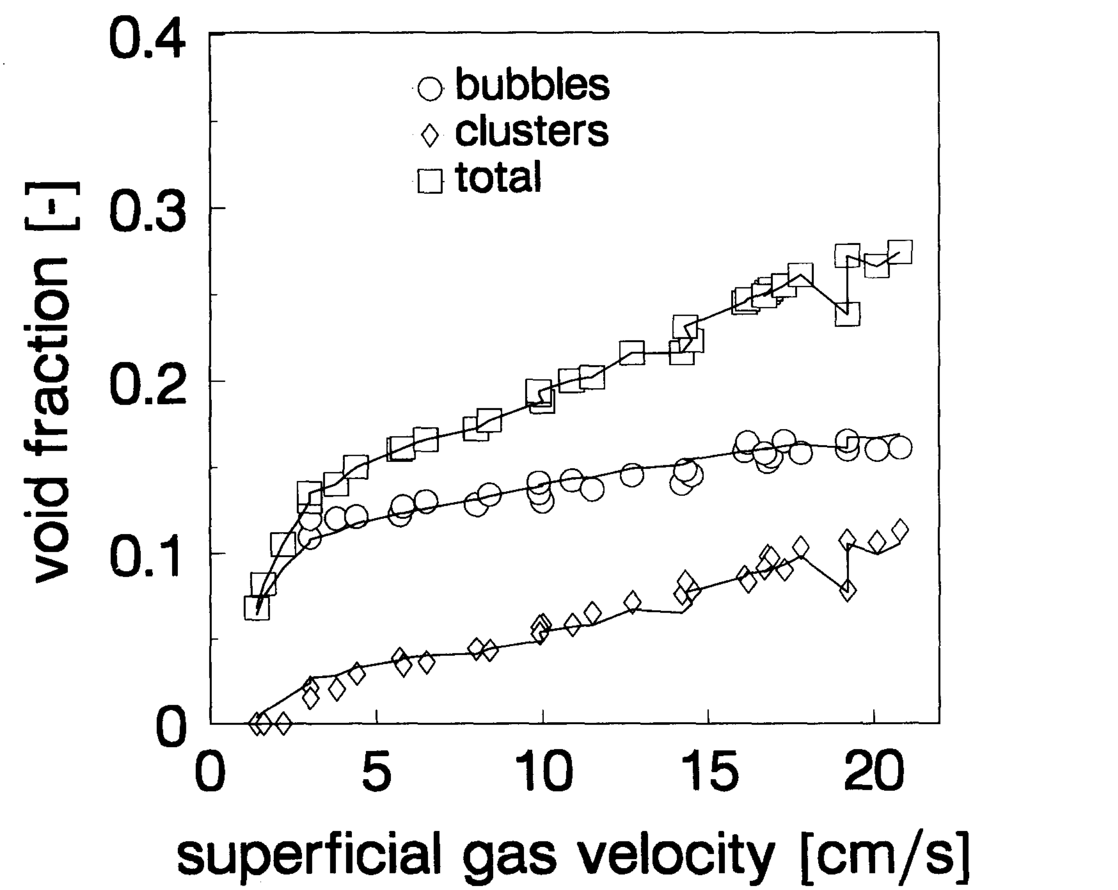


Figure 2. Measured (Schumpe & Grund 1986) and predicted void fractions of bubbles, clusters and total void fraction.

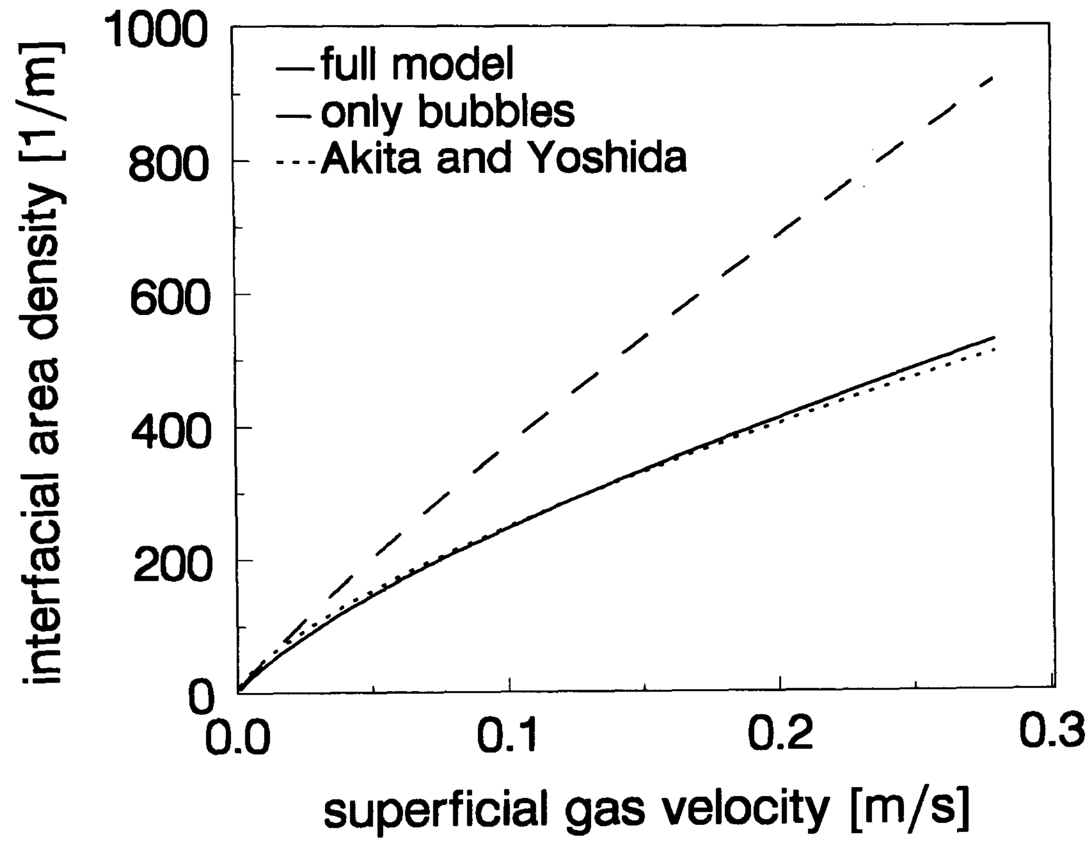


Figure 3. Interfacial area density from the analytical solution, the complete model including clusters, and from an empirical correlation, [2], by Akita & Yoshida (1974).

Schumpe & Grund (1986) did not measure the interfacial area density, the predicted interfacial area density is shown in figure 3. The analytical solution, given by [14], does not include bubble clusters. It is expected to give reasonable values only for small superficial gas velocities. The correlation of Akita & Yoshida (1974) is widely accepted. This correlation includes bubble-clusters and is in very good agreement with the interfacial area densities predicted by our model.

## 5. FIRST ORDER RELAXATION MODELS

### *5.1. Number density of bubbles and bubble clusters*

While [3] is a valid transport equation, it is often difficult to evaluate. Thus, let us develop a first order relaxation model for a bubbly flow containing bubble clusters. A perturbation analysis is made with the fully developed steady state reference. Only terms of up to first order are considered. The transport equation which is obtained may be written in terms of the linear difference of the variables from the steady state on the right hand side. Thus, it is called a relaxation model. A first order model will give accurate solutions only near the steady state. Thus, for example, describing a flow where one very large bubble is broken up into a swarm of small bubbles requires a full non-linear model.

First, we need to define the steady, full developed, state, which we will use as reference state. At any instant, we calculate the steady-state void fraction of bubbles outside of the clusters from the local value of the total void fraction. We then use table 1 to calculate the interfacial area densities for the steady-state.

Let us now consider the transient development of bubble clusters. The transport equations for the bubble clusters may be easily derived from [40]:

$$\frac{D_{\mathrm{Cl}}Cl[2]}{Dt} = r_2 N_b'''^2 - (r_5 + r_8 + 2r_{13}N_b''')Cl[2] + (r_8 + r_5)2Cl[3] \qquad [85]$$

and

$$\frac{D_{Cl}Cl[n]}{Dt} = (r_5 + r_8)(nCl[n+1] - (n-1)Cl[n]) + r_{13}N_b'''((n-1)Cl[n-1] - nCl[n]). \quad [86]$$

Let us observe how the number of clusters changes if the number of bubbles is suddenly decreased and then kept constant at a smaller value. The number of clusters therefore begins and ends at a steady-state. Figure 4 shows the difference of the number of bubble clusters in the two steady-states versus the nondimensional time

$$t^* = (r_5 + r_8)t. \quad [87]$$

For all practical parameter combinations, we find that there is one dominating eigenvalue of [85] and [86]. Indeed, the curves in figure 4 become nearly straight, parallel lines almost instantly. Thus, we have essentially a first order relaxation model.

We have evaluated the characteristic time corresponding to this eigenvalue numerically and have obtained:

$$\tau = \frac{1}{(r_5 + r_8)} \frac{1 + \chi}{(1 - \chi)}. \quad [88]$$

We have also obtained from these numerical calculations that the number of bubble clusters can be written as,

$$Cl[n, t, z] = a(t, z)s(t, z)^{n-2}, \quad [89]$$

as soon as the curves in figure 4 have become parallel. Note that

$$Cl[2, t, z] = a(t, z). \quad [90]$$

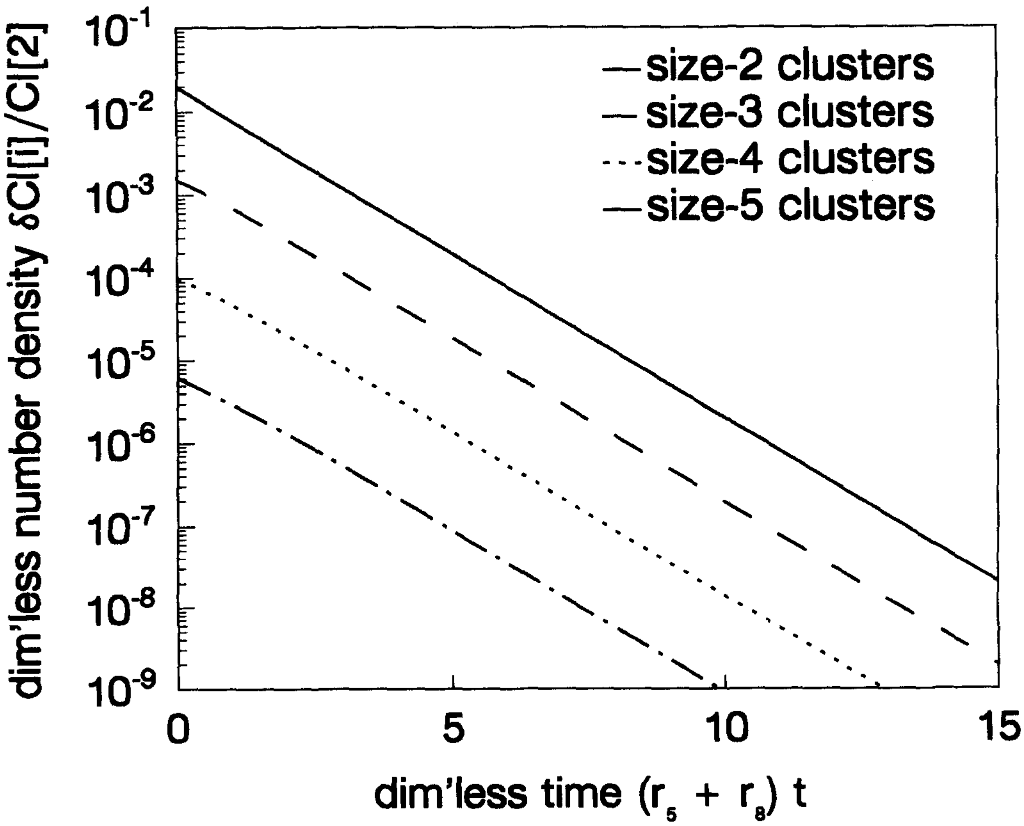


Figure 4. Difference from the steady-state values for the number of bubble cluster.

The steady-state values are

$$a(t \to \infty, z) \equiv a_\infty = \frac{r_2 N_b'''^2}{r_5 + r_8} \tag{91}$$

and

$$s(t \to \infty, z) = \chi. \tag{92}$$

We look for a relation between the deviation of $a$ from its steady-state value,

$$\delta a = a - a_\infty \tag{93}$$

and the deviation of $s$ from its steady-state value

$$\delta s = s - \chi. \tag{94}$$

We obtain the transport equation for the total number of clusters from [45]:

$$\frac{\partial \sum Cl}{\partial t} + \nabla \cdot \left[ \mathrm{u}_{Cl} \sum Cl \right] = r_2 N_b'''^2 - (r_5 + r_8)a. \tag{95}$$

The first order approximation is

$$\frac{\partial \sum Cl}{\partial t} + \nabla \cdot \left[ \mathrm{u}_{Cl} \sum Cl \right] = 2 r_2 N_b''' \delta N_b''' - (r_5 + r_8)\delta a. \tag{96}$$

Using [89],

$$Cl[i] \approx a s^{i-2}, \tag{97}$$

we have,

$$\sum Cl \approx \frac{a}{1 - s} \tag{98}$$

and from a first order approximation (recall that $\chi$ is the steady-state value of $s$):

$$\delta \sum Cl = \frac{1}{1 - \chi} \delta a + a \delta s \frac{1}{(1 - \chi)^2}. \tag{99}$$

Since we know the relaxation time from [88], we may conclude that, as soon as the curves in figure 4 become parallel (we remark that the number of bubbles per unit volume, $N_b'''$, stays constant after the initial jump):

$$\underbrace{-(r_5 + r_8) \frac{1 - \chi}{1 + \chi}}_{1/\tau} \left( \frac{\delta a}{1 - \chi} + \frac{a \delta s}{(1 - \chi)^2} \right) = -(r_5 + r_8)\delta a \tag{100}$$

and, therefore,

$$a \delta s = \delta a \chi (1 - \chi). \tag{101}$$

We will make use of these equations later on.

### 5.2. Bubble volume

The transport equation for the gas volume in size-2 clusters may be derived from [47] as,

$$\frac{\partial}{\partial t}(Cl[2]2\bar{v}_{Cl}[2]) + \nabla \cdot [u_{Cl}(Cl[2]2\bar{v}_{Cl}[2])] = 2r_2 N_b'''^2 \bar{v}_b + 2r_5(3Cl[3]\bar{v}_{Cl}[3] - Cl[2]\bar{v}_{Cl}[2])$$
$$+ 2r_8(2Cl[3]\bar{v}_{Cl}[3] - Cl[2]\bar{v}_{Cl}[2]) - 2r_{13}N_b''' Cl[2]v_{Cl}[2]. \quad [102]$$

We subtract [47], multiplied on both sides with $2\bar{v}_{Cl}[2]$, from [102] and obtain:

$$2Cl[2]\left(\frac{\partial v_{Cl}[2]}{\partial t} + \nabla \cdot [u_{Cl}v_{Cl}[2]]\right) = r_2 N_b'''^2(\bar{v}_b - \bar{v}_{Cl}[2]) + r_5 Cl[3]\bar{v}_{Cl}[3]$$
$$+ 2(r_5 + r_8)Cl[3](\bar{v}_{Cl}[3] - \bar{v}_{Cl}[2]). \quad [103]$$

The transport equation for the gas volume in size-$n$ clusters may be derived from [48],

$$\frac{\partial}{\partial t}(Cl[n]n\bar{v}_{Cl}[n]) + \nabla \cdot [\bar{u}_{Cl}(Cl[n]n\bar{v}_{Cl}[n])] = r_5(nCl[n+1](n+1)\bar{v}_{Cl}[n+1]$$
$$-(n-1)Cl[n]n\bar{v}_{Cl}[n])$$
$$+r_8(nCl[n+1]n\bar{v}_{Cl}[n+1]$$
$$-(n-1)Cl[n]n\bar{v}_{Cl}[n])$$
$$-r_{13}N_b'''(nCl[n]n\bar{v}_{Cl}[n]$$
$$-(n-1)Cl[n-1]$$
$$\times((n-1)\bar{v}_{Cl}[n-1] + \bar{v}_b)). \quad [104]$$

We subtract [48], multiplied on both sides with $n\bar{v}_{Cl}[n]$, from [104] and obtain:

$$nCl[n]\left(\frac{\partial \bar{v}_{Cl}[n]}{\partial t} + \nabla \cdot [u_{Cl}\bar{v}_{Cl}[n]]\right) = r_5 nCl[n+1]\bar{v}_{Cl}[n+1] + (r_5 + r_8)n^2 Cl[n+1](\bar{v}_{Cl}[n+1] - \bar{v}_{Cl}[n])$$
$$+ r_{13}N_b'''(n-1)Cl[n-1]$$
$$\times((n-1)\bar{v}_{Cl}[n-1] + \bar{v}_b - n\bar{v}_{Cl}[n]). \quad [105]$$

Let us observe how the average bubble volume inside the clusters changes, if we keep the number of bubbles outside of clusters constant, but suddenly decrease the average volume of bubbles outside the clusters. From [85] and [86], the number of clusters then remains constant. The average bubble volume minus its steady state value is presented for several sizes of clusters in figure 5. We observe that the average bubble volumes in all clusters change in the same way. This holds also, even though only approximately, if we change the number of bubbles outside the clusters instead of the bubble volume. The average bubble volumes are therefore the same in all clusters, even for transient processes. We also find from the numerical calculation that

$$\delta\left(\sum_{i=2}^{\infty} Cl[i]\bar{v}_{Cl}\right) \approx \delta\left(\sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}\right)\frac{\sum_{i=2}^{\infty} Cl[i]}{\sum_{i=2}^{\infty} iCl[i]} \quad [106]$$

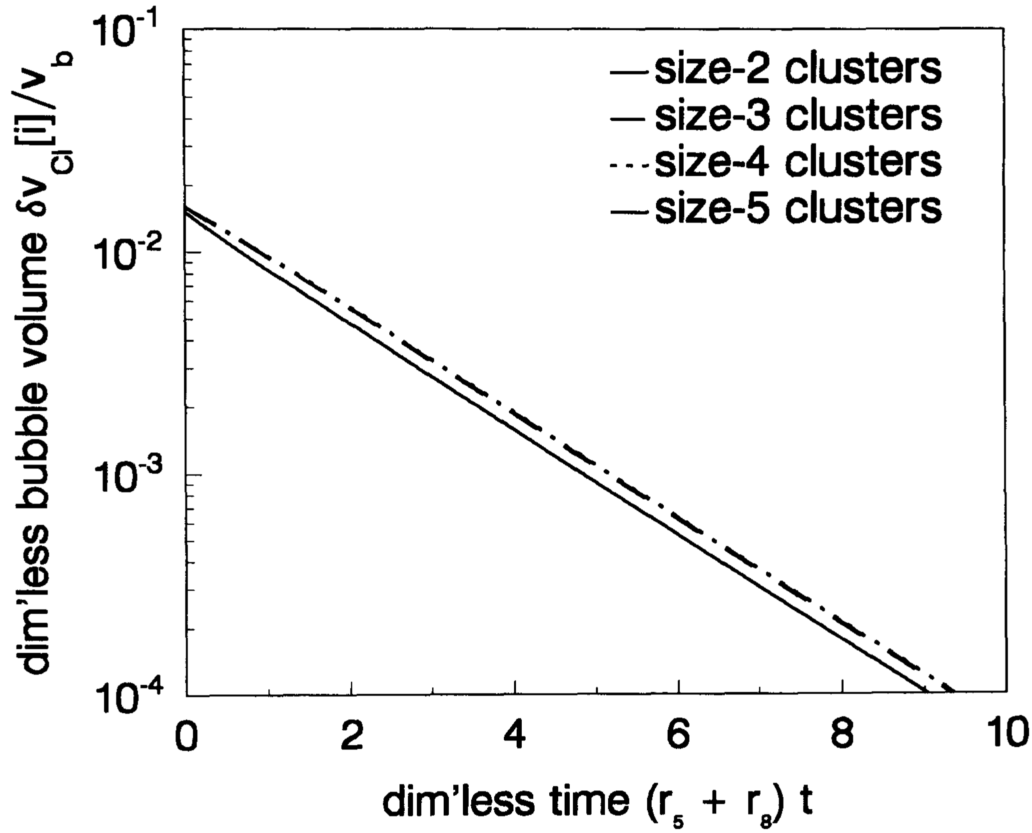


Figure 5. Differences from the steady-state values for average bubble volume in several bubble clusters.

and

$$\delta(Cl[2]\bar{v}_{Cl}) \approx \delta\left(\sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}\right)\frac{Cl[2]}{\sum_{i=2}^{\infty} iCl[i]}, \tag{107}$$

give good approximations. Using [45] and [46], we find from [106]:

$$\delta\left(\sum_{i=2}^{\infty} Cl[i]\bar{v}_{Cl}\right) \approx \delta\left(\sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}\right)\frac{1-\chi}{2-\chi} \tag{108}$$

and

$$\delta(Cl[2]\bar{v}_{Cl}) \approx \delta\left(\sum_{i=2}^{\infty} iCl[i]\bar{v}_{Cl}\right)\frac{(1-\chi)^2}{2-\chi}. \tag{109}$$

We will make use of these equations later on.

### 5.3. *Interfacial area density*

Our goal here is to develop a first order relaxation model for the transient interfacial area density of the bubbles (see [13] and [82]), which is at each instant,

$$A_{ib}''' = (36\pi)^{1/3}\int_0^{\infty} u^{2/3}\,\frac{N_b'''}{\bar{v}_b}\,e^{-u/\bar{v}_b}\,du = (36\pi)^{1/3}N_b'''\bar{v}_b^{2/3}\tfrac{2}{3}\Gamma(\tfrac{2}{3}) \tag{110}$$

thus,

$$A'''_{ib} = 4.37 N'''_b \bar{v}_b^{2/3} \qquad [111]$$

and for the bubble clusters at each time $t$,

$$A'''_{iCl} = 4.37 \sum_{i=2}^{\infty} (iCl[i]) \bar{v}_{Cl}^{2/3}. \qquad [112]$$

Kataoka *et al.* (1986) proved that the ergodic theorem holds for the interfacial area density, thus we may use either the time, space or ensemble averaged value for the interfacial area density and still end up with the same equations, [111] and [112]. We may derive the transport equations of interest for the interfacial area densities from the transport equations for the void fractions and for the number of bubbles and bubble clusters, from [111]:

$$\delta A'''_{ib} = A'''_{ib} \left( \frac{2}{3} \frac{\delta (N'''_b \bar{v}_b)}{N'''_b \bar{v}_b} + \frac{1}{3} \frac{N'''_b}{N'''_b} \right)$$

$$= A'''_{ib} \left( \frac{2}{3} \frac{\delta \alpha_b}{\alpha_b} + \frac{1}{3} \frac{\delta N'''_b}{N'''_b} \right) \qquad [113]$$

and for the bubble clusters, from [112]:

$$\delta A'''_{iCl} = A'''_{iCl} \quad \frac{2}{3} \frac{\delta\left(\sum iCl \bar{v}_{Cl}\right)}{\sum iCl \bar{v}_{Cl}} + \frac{1}{3} \frac{\delta \sum iCl}{\sum iCl}$$

$$= A'''_{iCl} \quad \frac{2}{3} \frac{d\alpha_{Cl}}{\alpha_{Cl}} + \frac{1}{3} \frac{d \sum iCl}{\sum iCl} \quad , \qquad [114]$$

where $\Sigma\, iCl$ is an abbreviation for

$$\sum iCl = \sum_{i=2}^{\infty} iCl[i]. \qquad [115]$$

The transport equations for the interfacial area densities are therefore,

$$\frac{\partial A'''_{ib}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b A'''_{ib}] = \tfrac{1}{3} A'''_{ib} \left[ \frac{2}{\alpha_b} \left( \frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] \right) + \frac{1}{N'''_b} \left( \frac{\partial N'''_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b N'''_b] \right) \right] \qquad [116]$$

and

$$\frac{\partial A'''_{iCl}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_{Cl} A'''_{iCl}] = \tfrac{1}{3} A'''_{iCl} \left[ \frac{2}{\alpha_{Cl}} \left( \frac{\partial \alpha_{Cl}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_{Cl} \alpha_{Cl}] \right) + \frac{1}{\sum iCl} \left( \frac{\partial \sum iCl}{\partial t} + \nabla \cdot \left[\bar{\mathbf{u}}_{Cl} \sum iCl\right] \right) \right], \qquad [117]$$

where $\bar{\mathbf{u}}_b$ is the average velocity of the bubbles, and $\bar{\mathbf{u}}_{Cl}$ is the average velocity of the bubble clusters. The transport equation for the void fraction of the bubbles can be obtained by setting the rate of change of bubble volume fraction equal to the sum of the sources and sinks by [47], which describes the steady-state,

$$\frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] = \underbrace{-r_2 N_b'''2\bar{v}_b}_{\text{Grouping}} + \underbrace{(r_5 + r_8)Cl[2]2\bar{v}_{Cl}}_{\text{Coalescence and removal}}$$

$$\underbrace{- r_{13} N_b''' \sum_{i=2}^{\infty} iCl\bar{v}_b}_{\text{Uptake}} + \underbrace{r_8 \sum_{i=3}^{\infty} (i-1)Cl[i]\bar{v}_{Cl}}_{\text{Removal}}. \quad [118]$$

Since we are interested in a first order relaxation model, we only regard relatively small disturbances from the steady-state. If $\delta$ signifies a perturbation about the steady-state value, we obtain from a Taylor expansion, canceling out the terms from the steady-state equation, [48],

$$\frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] = -2r_2\alpha_b \delta N_b''' - 2r_2 N_b''' \delta\alpha_b + 2(r_5 + r_8)\delta(a\bar{v}_{Cl})$$

$$- r_{13}\delta(\alpha_b)\sum iCl - r_{13}\alpha_b\delta\left(\sum iCl\right) + r_8\delta(\alpha_{Cl}) - r_8\delta\left(\bar{v}_{Cl}\sum Cl\right), \quad [119]$$

where

$$\sum Cl = \sum_{i=2}^{\infty} Cl[i] \quad [120]$$

and, from [89],

$$Cl[2] = a. \quad [121]$$

The transport equation for the number of bubbles becomes from [47] and [48],

$$\frac{\partial N_b'''}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b N_b'''] = \tfrac{1}{2} r_1 \bar{v}_b N_b''' + r_8 \sum_{i=3}^{\infty} (i-1)Cl[i] + r_8 2Cl[2] + r_5 Cl[2]$$

$$- 2r_2 N_b'''^2 - r_{13} N_b''' \sum_{i=2}^{\infty} iCl[i]. \quad [122]$$

Thus, the first order approximation is,

$$\frac{\partial N_b'''}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b N_b'''] = \tfrac{1}{2} r_1 \delta\alpha_b + r_8\delta\left(\sum iCl\right) - r_8\delta\left(\sum Cl\right)$$

$$+ (r_5 + r_8)\delta a - 4r_2 N_b''' \delta N_b''' - r_{13}\delta N_b''' \sum iCl - r_{13} N_b''' \delta\left(\sum iCl\right). \quad [123]$$

This transport equation, and therefore also the first order approximation for the void fraction of the bubble clusters, differs from [119] only by the first term on the right hand side, as it has an opposite sign:

$$\frac{\partial \alpha_{Cl}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_{Cl} \alpha_{Cl}] = 2r_2\alpha_b \delta N_b''' + 2r_2 N_b''' \delta\alpha_b - 2(r_5 + r_8)\delta(a\bar{v}_{Cl})$$

$$+ r_{13}\delta(\alpha_b)\sum iCl + r_{13}\alpha_b\delta\left(\sum iCl\right) - r_8\delta(\alpha_{Cl}) + r_8\delta\left(\bar{v}_{Cl}\sum Cl\right). \quad [124]$$

We now use the transport equation for the number of clusters, $\Sigma\, Cl$, from [96] instead of the number of bubbles in the clusters, $\Sigma\, iCl$, as it will prove useful later: Thus we find from [101] and [99],

$$\delta \sum Cl = \delta a \frac{1+\chi}{1-\chi}. \qquad [125]$$

Therefore,

$$\frac{\partial \sum Cl}{\partial t} + \nabla \cdot \left[ \bar{\mathbf{u}}_{Cl} \sum Cl \right] = 2r_2 N_b''' \delta N_b''' - (r_5 + r_8) \frac{1-\chi}{1+\chi} \delta \sum Cl. \qquad [126]$$

We are interested in a transport equation for the number of bubbles in bubble clusters, $\Sigma\, iCl$. To this end, [89] implies,

$$\sum iCl \approx \sum_{i=2}^{\infty} ias^{i-2} = a \frac{2-s}{(1-s)^2}. \qquad [127]$$

Therefore,

$$\delta \sum iCl \approx \delta a \frac{2-\chi}{(1-\chi)^2} + a\delta s \frac{3-\chi}{(1-\chi)^3}, \qquad [128]$$

which from [101] is

$$\delta \sum iCl = \delta a \frac{3-(1-\chi)^2}{(1-\chi)^2}. \qquad [129]$$

This is an accurate solution for large times, but it underestimates the number of bubbles in clusters, $\Sigma\, iCl$, for small times. From our numerical calculations, we find the approximation,

$$\delta \sum iCl = 2\delta a \frac{1+\chi}{(1-\chi)^2}, \qquad [130]$$

which is a compromise between the analytical solution for large times, [129], and the solution for small times, which may be found from the old steady-state values. We find from [125] as an accurate solution for large times,

$$\delta \sum iCl = \frac{3-(1-\chi)^2}{1-\chi^2} \delta \sum Cl, \qquad [131]$$

which we approximate for our numerical calculation, as done with [130],

$$\delta \sum iCl = 4 \frac{1+\chi}{2-\chi} \delta \sum Cl. \qquad [132]$$

The transport equation for the number of bubbles in the bubble clusters is therefore, from [126],

$$\frac{\partial \sum iCl}{\partial t} + \nabla \cdot \left[ \bar{\mathbf{u}}_{Cl} \sum iCl \right] = 8r_2 N_b''' \frac{1+\chi}{2-\chi} \delta N_b''' - (r_5 + r_8) \frac{1-\chi}{1+\chi} \delta \sum iCl. \qquad [133]$$

The first order transport equation for the interfacial area density of the bubbles, [116], becomes,

$$\frac{3}{A'''_{ib}}\left(\frac{\partial A'''_{ib}}{\partial t} + \nabla\cdot[\bar{\mathbf{u}}_b A'''_{ib}]\right) = -4r_2\delta N'''_b - 4r_2 N'''_b\frac{\delta\alpha_b}{\alpha_b} + 4(r_5+r_8)\frac{\delta(a\bar{v}_{Cl})}{\alpha_b}$$

$$-2r_{13}\frac{\delta\alpha_b}{\alpha_b}\sum iCl - 2r_{13}\delta\sum iCl + 2r_8\frac{\delta\alpha_{Cl}}{\alpha_b} - 2r_8\frac{\delta\bar{v}_{Cl}\sum Cl}{\alpha_b}$$

$$+\tfrac{1}{2}r_1\frac{\delta\alpha_b}{N'''_b} + r_8\frac{\delta\sum iCl}{N'''_b} - r_8\frac{\delta\sum Cl}{N'''_b} + (r_5+r_8)\frac{\delta a}{N'''_b} - 4r_2\delta N'''_b$$

$$-r_{13}\frac{\delta N'''_b}{N'''_b}\sum iCl - r_{13}\delta\sum iCl. \qquad [134]$$

Let us now convert the right hand side of [134] into a form where only disturbances of $A'''_{ib}$, $A'''_{iCl}$, $\alpha_b$ and $\alpha_{Cl}$ occur: the disturbance in the number of bubbles is, from [113]:

$$\frac{\delta N'''_b}{N'''_b} = 3\frac{\delta A'''_{ib}}{A'''_{ib}} - 2\frac{\delta\alpha_b}{\alpha_b}. \qquad [135]$$

The perturbation of $(a\bar{v}_{Cl})$ is, from [109],

$$\delta(a\bar{v}_{Cl}) = \frac{(1-\chi)^2}{2-\chi}\delta\alpha_{Cl}. \qquad [136]$$

The perturbation in the number of bubbles in the bubble clusters is, from [114],

$$\frac{\delta\sum iCl}{\sum iCl} = 3\frac{A'''_{iCl}}{A'''_{iCl}} - 2\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}. \qquad [137]$$

The perturbation in $(\bar{v}_{Cl}\,\Sigma\, Cl)$, is from [106],

$$\delta\left(\bar{v}_{Cl}\sum Cl\right) = \frac{1-\chi}{2-\chi}\delta\alpha_{Cl}. \qquad [138]$$

The perturbation in $a$ is from [130],

$$\delta a = \frac{(1-\chi)^2}{2(1+\chi)}\delta\sum iCl = \frac{(1-\chi)^2}{2(1+\chi)}\frac{1}{\bar{v}_{Cl}}\left(3\alpha_{Cl}\frac{\delta A'''_{iCl}}{A'''_{iCl}} - 2\delta\alpha_{Cl}\right). \qquad [139]$$

Combining these expressions and using [134]:

$$\frac{3}{A'''_{ib}}\left(\frac{A'''_{ib}}{\partial t} + \nabla\cdot[\bar{\mathbf{u}}_b A'''_{ib}]\right) = -12r_2\frac{\alpha_b}{\bar{v}_b}\frac{A'''_{ib}}{A'''_{ib}} + 8\frac{r_2}{\bar{v}_b}\delta\alpha_b - 4\frac{r_2}{\bar{v}_b}\delta\alpha_b + 4(r_5+r_8)\frac{(1-\chi)^2}{2-\chi}\frac{\delta\alpha_{Cl}}{\alpha_b}$$

$$-2r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}\alpha_b}\delta\alpha_b - 6r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\frac{\delta A'''_{iCl}}{A'''_{iCl}} + 4r_{13}\frac{\delta\alpha_{Cl}}{\bar{v}_{Cl}} + 2r_8\frac{\delta\alpha_{Cl}}{\alpha_b}$$

$$-2r_8\frac{1-\chi}{2-\chi}\frac{\delta\alpha_{Cl}}{\alpha_b} + \tfrac{1}{2}\frac{r_1\bar{v}_b}{\alpha_b}\delta\alpha_b$$

$$+r_8\frac{\bar{v}_b}{\alpha_b}\left(3\frac{\delta A'''_{iCl}}{A'''_{iCl}}-2\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}\right)\left(1-\frac{2-\chi}{4(1+\chi)}\right)\frac{\alpha_{Cl}}{\bar{v}_{Cl}}$$

$$+(r_5+r_8)\frac{(1-\chi)^2}{2(1+\chi)}\frac{\bar{v}_b}{\bar{v}_{Cl}}\left(3\frac{\alpha_{Cl}}{\alpha_b}\frac{A'''_{iCl}}{A'''_{iCl}}-2\frac{\delta\alpha_{Cl}}{\alpha_b}\right)$$

$$-12r_2\frac{\alpha_b}{\bar{v}_b}\frac{\delta A'''_{ib}}{A'''_{ib}}+8r_2\frac{\delta\alpha_b}{\bar{v}_b}-r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\left(3\frac{A'''_{ib}}{A'''_{ib}}-2\frac{\delta\alpha_b}{\alpha_b}\right)$$

$$-3r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\frac{\delta A'''_{iCl}}{A'''_{iCl}}+2r_{13}\frac{\delta\alpha_{Cl}}{\bar{v}_{Cl}}. \qquad [140]$$

We may write this equation as a first order relaxation model, using the fact that the perturbation in the void fractions of the bubbles and bubble clusters from their steady-state value are of opposite sign but have equal value. Therefore,

$$\frac{\partial A'''_{ib}}{\partial t}+\nabla\cdot[\bar{\mathbf{u}}_b A'''_{ib}]=a_{11}(A'''_{ib}-A'''_{ib\infty})+a_{12}\frac{A'''_{ib\infty}}{A'''_{iCl\infty}}(A'''_{iCl}-A'''_{iCl\infty})+a_{13}\frac{A'''_{ib\infty}}{\alpha_{b\infty}}(\alpha_b-\alpha_{b\infty}), \qquad [141]$$

where

$$a_{11}=-8r_2\frac{\alpha_b}{\bar{v}_b}-r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}, \qquad [142]$$

$$a_{12}=-2r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}+(r_5+r_8)\frac{\alpha_{Cl}\bar{v}_b}{\alpha_b\bar{v}_{Cl}}\frac{(1-\chi)^2}{2(1+\chi)}+r_8\frac{\alpha_{Cl}\bar{v}_b}{\alpha_b\bar{v}_{Cl}}\left(1-\frac{2-\chi}{4(1+\chi)}\right)-r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}, \qquad [143]$$

or, using [44] to eliminate $(r_5+r_8)$,

$$a_{12}=r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\left(-3+\frac{1}{4\chi(1+\chi)}\left[2(1-\chi)^2+\frac{r_8}{r_5+r_8}(2+5\chi)\right]\right) \qquad [144]$$

and

$$a_{13}=\frac{r_2}{3\bar{v}_b}\alpha_b(8-4)-\tfrac{4}{3}(r_5+r_8)\frac{(1-\chi)^2}{2-\chi}-\tfrac{2}{3}r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}$$

$$-\tfrac{4}{3}r_{13}\frac{\alpha_b}{\bar{v}_{Cl}}-\tfrac{2}{3}r_8+\tfrac{2}{3}r_8\frac{1-\chi}{2-\chi}+\tfrac{1}{6}r_1\bar{v}_b+\tfrac{2}{3}r_8\frac{\bar{v}_b}{\bar{v}_{Cl}}$$

$$-\tfrac{2}{3}r_8\frac{\bar{v}_b}{\bar{v}_{Cl}}\frac{2-\chi}{4(1+\chi)}+\tfrac{2}{3}(r_5+r_8)\frac{\bar{v}_b}{\bar{v}_{Cl}}\frac{(1-\chi)^2}{2(1+\chi)}+\tfrac{8}{3}r_2\frac{\alpha_b}{\bar{v}_b}+\tfrac{2}{3}r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}} \qquad [145]$$

or, using [44] to eliminate $(r_5+r_8)$,

$$a_{13}=\tfrac{1}{6}r_1\bar{v}_b+4r_2\frac{\alpha_b}{\bar{v}_b}-\tfrac{2}{3}r_{13}\frac{\alpha_b}{\bar{v}_b}\frac{1}{\chi(2-\chi)}\left[2(1-\chi)^2+\frac{r_8}{r_5+r_8}\right]+\tfrac{2}{3}a_{12}\frac{\alpha_b}{\alpha_{Cl}}. \qquad [146]$$

Similarly, the transport equation for the interfacial area density of the bubble clusters, [117], becomes:

$$\frac{3}{A'''_{iCl}}\left(\frac{\partial A'''_{iCl}}{\partial t}+\nabla\cdot[\bar{\mathbf{u}}_{Cl}A'''_{iCl}]\right)=4r_2\frac{\alpha_b}{\alpha_{Cl}}\delta N'''_b+4r_2N'''_b\frac{\delta\alpha_b}{\alpha_{Cl}}-4(r_5+r_8)\frac{\delta(a\bar{v}_{Cl})}{\alpha_{Cl}}$$

$$+2r_{13}\frac{\delta\alpha_b}{\alpha_{Cl}}\sum iCl+2r_{13}\frac{\alpha_b}{\alpha_{Cl}}\delta\sum iCl-2r_8\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}$$

$$+2\frac{r_8}{\alpha_{Cl}}\delta\left(\bar{v}_{Cl}\sum Cl\right)+8r_2\frac{\alpha_b\bar{v}_{Cl}}{\alpha_{Cl}\bar{v}_b}\frac{(1+\chi)}{2-\chi}\delta N_b'''$$

$$-(r_5+r_8)\frac{1-\chi}{1+\chi}\frac{\delta\sum iCl}{\sum iCl}. \qquad [147]$$

We use [135], [136] and [114] and obtain,

$$\frac{3}{A_{iCl}'''}\left(\frac{\partial A_{iCl}'''}{\partial t}+\nabla\cdot[\bar{\mathbf{u}}_{Cl}A_{iCl}''']\right)=4r_2\frac{\alpha_b}{\alpha_{Cl}}\left(3\frac{\delta A_{ib}'''}{A_{ib}'''}-2\frac{\delta\alpha_b}{\alpha_b}\right)\frac{\alpha_b}{\bar{v}_b}+4r_2\frac{\alpha_b}{\bar{v}_b}\frac{\delta\alpha_b}{\alpha_{Cl}}$$

$$-4(r_5+r_8)\frac{(1-\chi)^2}{2-\chi}\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}+2\frac{r_{13}}{\bar{v}_{Cl}}\delta\alpha_b$$

$$+2r_{13}\frac{\alpha_b}{\bar{v}_{Cl}}\left(3\frac{A_{iCl}'''}{A_{iCl}'''}-2\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}\right)-2r_8\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}+2r_8\frac{1-\chi}{2-\chi}\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}$$

$$+8r_2\frac{\alpha_b^2\bar{v}_{Cl}}{\alpha_{Cl}\bar{v}_b^2}\frac{(1+\chi)}{2-\chi}\left(3\frac{\delta A_{ib}'''}{A_{ib}'''}-2\frac{\delta\alpha_b}{\alpha_b}\right)$$

$$-(r_5+r_8)\frac{1-\chi}{1+\chi}\left(3\frac{\delta A_{iCl}'''}{A_{iCl}'''}-2\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}\right). \qquad [148]$$

We may write this in the form of a first order relaxation model as:

$$\frac{\partial A_{iCl}'''}{\partial t}+\nabla\cdot[\bar{\mathbf{u}}_{Cl}A_{iCl}''']=a_{21}\frac{A_{iCl\infty}'''}{A_{ib\infty}'''}(A_{ib}'''-A_{ib\infty}''')+a_{22}(A_{iCl}'''-A_{iCl\infty}''')+a_{23}\frac{A_{iCl\infty}'''}{\alpha_{b\infty}}(\alpha_b-\alpha_{b\infty}), \qquad [149]$$

where

$$a_{21}=4r_2\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}\left(1+2\frac{\bar{v}_{Cl}}{\bar{v}_b}\frac{(1+\chi)}{(2-\chi)}\right), \qquad [150]$$

$$a_{22}=2r_{13}\frac{\alpha_b}{\bar{v}_{Cl}}-(r_5+r_8)\frac{1-\chi}{1+\chi}, \qquad [151]$$

or, using [44] to eliminate $(r_5+r_8)$,

$$a_{22}=2r_{13}\frac{\alpha_b}{\bar{v}_{Cl}}-r_{13}\frac{\alpha_b}{\bar{v}_b}\frac{1-\chi}{\chi(1+\chi)} \qquad [152]$$

and

$$a_{23}=-\tfrac{8}{3}r_2\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}+\tfrac{4}{3}r_2\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}+\tfrac{4}{3}(r_5+r_8)\frac{(1-\chi)^2}{2-\chi}\frac{\alpha_b}{\alpha_{Cl}}+\tfrac{2}{3}\frac{r_{13}\alpha_b}{\bar{v}_{Cl}}+\tfrac{4}{3}r_{13}\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_{Cl}}+\tfrac{2}{3}r_8\frac{\alpha_b}{\alpha_{Cl}}-\tfrac{2}{3}r_8\frac{1-\chi}{2-\chi}\frac{\alpha_b}{\alpha_{Cl}}$$

$$-\tfrac{16}{3}r_2\frac{\alpha_b^2\bar{v}_{Cl}}{\alpha_{Cl}\bar{v}_b^2}\frac{(1+\chi)}{(2-\chi)}-(r_5+r_8)\frac{1-\chi}{1+\chi}\frac{2}{3}\frac{\alpha_b}{\alpha_{Cl}}=-\tfrac{4}{3}r_2\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}\left(1+4\frac{\bar{v}_{Cl}}{\bar{v}_b}\frac{(1+x)}{(2-\chi)}\right)$$

$$+\tfrac{2}{3}r_{13}\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}\left[\frac{(1-2\chi)(1-\chi)}{(2-\chi)(1+\chi)}+\frac{r_8/(r_5+r_8)}{\chi(2-\chi)}\right]+\tfrac{2}{3}r_{13}\frac{\alpha_b}{\bar{v}_{Cl}}\left(1+2\frac{\alpha_b}{\alpha_{Cl}}\right). \qquad [153]$$

We also need the transport equations for the total void fraction of the two-phase system,

$$\frac{\partial\alpha}{\partial t}+\nabla\cdot[\bar{\mathbf{u}}_b\alpha_b]+\nabla\cdot[\bar{\mathbf{u}}_{Cl}\alpha\,\mathrm{Cl}]=0 \qquad [154]$$

and for the void fraction of the bubbles outside of clusters, from [119]:

$$\frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] = -\alpha_b \frac{\partial u_b}{\partial z} - 2r_2 \frac{\alpha_b^2}{\bar{v}_b}\left(3\frac{\delta A'''_{ib}}{A'''_{ib}} - 2\frac{\delta \alpha_b}{\alpha_b}\right) - 2r_2 \frac{\alpha_b}{\bar{v}_b}\delta\alpha_b + 2(r_5 + r_8)\frac{(1-\chi)^2}{2-\chi}\delta\alpha_{Cl}$$

$$-r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\delta\alpha_b - r_{13}\alpha_b\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\left(3\frac{\delta A'''_{iCl}}{A'''_{iCl}} - 2\frac{\delta\alpha_{Cl}}{\alpha_{Cl}}\right) + r_8\delta\alpha_{Cl} - r_8\frac{1-\chi}{2-\chi}\delta\alpha_{Cl}. \quad [155]$$

We can rewrite this equation in the form of a first order relaxation model,

$$\frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] = a_{31}\frac{\alpha_{b\infty}}{A'''_{ib\infty}}(A'''_{ib} - A'''_{ib\infty}) + a_{32}\frac{\alpha_{b\infty}}{A'''_{iCl\infty}}(A'''_{iCl} - A'''_{iCl\infty}) + a_{33}(\alpha_b - \alpha_{b\infty}), \quad [156]$$

where

$$a_{31} = -6r_2\frac{\alpha_b}{\bar{v}_b}, \quad [157]$$

$$a_{32} = -3r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}} \quad [158]$$

and

$$a_{33} = 4r_2\frac{\alpha_b}{\bar{v}_b} - 2r_2\frac{\alpha_b}{\bar{v}_b} - 2(r_5 + r_8)\frac{(1-\chi)^2}{2-\chi} - r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}} - 2r_{13}\frac{\alpha_b}{\bar{v}_{Cl}} - r_8 + r_8\frac{1-\chi}{2-\chi}$$

$$= 2r_2\frac{\alpha_b}{\bar{v}_b} - r_{13}\frac{\alpha_b}{\bar{v}_b}\left(\frac{2(1-\chi)^2 + r_8/(r_5+r_8)}{\chi(2-\chi)}\right) - r_{13}\frac{1+\alpha_b}{\bar{v}_{Cl}}. \quad [159]$$

Further terms may be introduced into this equation. Compressibility and phase change effects are introduced in the next chapter. Turbulent bubble diffusion also gives an additional term which we do not include in the present paper.

### 5.4. *Compressibility and phase change effects*

Kelly (1993) considered two additional source terms for the interfacial area density; one due to the compressibility (C) of the gas, and the other due to possible vapor generation ($g$) near the wall. If we include these source terms in [114], [149], [156], we obtain:

$$\frac{\partial A'''_{ib}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b A'''_{ib}] = a_{11}(A'''_{ib} - A'''_{ib\infty}) + a_{12}\frac{A'''_{ib\infty}}{A'''_{iCl\infty}}(A'''_{iCl} - A'''_{iCl\infty})$$

$$+ a_{13}\frac{A'''_{ib\infty}}{\alpha_{b\infty}}(\alpha_b - \alpha_{b\infty}) + \left(\frac{dA'''_{ib}}{dt}\right)_C + \left(\frac{dA'''_{ib}}{dt}\right)_r, \quad [160]$$

$$\frac{\partial A'''_{iCl}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_{Cl} A'''_{iCl}] = a_{21}\frac{A'''_{iCl\infty}}{A'''_{ib\infty}}(A'''_{ib} - A'''_{ib\infty}) + a_{22}(A'''_{iCl} - A'''_{iCl\infty})$$

$$+ a_{23}\frac{A'''_{iCl\infty}}{\alpha_{b\infty}}(\alpha_b - \alpha_{b\infty}) + \left(\frac{dA'''_{iCl}}{dt}\right)_C + \left(\frac{dA'''_{iCl}}{dt}\right)_r \quad [161]$$

and

$$\frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] = \alpha_{31}\frac{\alpha_{b\infty}}{A'''_{ib\infty}}(A'''_{ib} - A'''_{ib\infty}) + a_{32}\frac{\alpha_{b\infty}}{A'''_{iCl\infty}}(A'''_{iCl} - A'''_{iCl\infty})$$

$$+ a_{33}(\alpha_b - \alpha_{b\infty}) + \left(\frac{d\alpha_b}{dt}\right)_C + \left(\frac{d\alpha_b}{dt}\right)_r. \quad [162]$$

Kelly (1993) proposed for the source term due to the compressibility of the gas,

$$\left(\frac{\mathrm{d}A'''_{ib}}{\mathrm{d}t}\right)_C = \alpha_b \rho_G \left(\frac{\mathrm{d}a_{ib}}{\mathrm{d}v_b}\right)\left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{u}_b/\rho_G]\right), \qquad [163]$$

where $(\mathrm{d}a_{ib}/\mathrm{d}v_b)$ is the derivative of the interfacial surface area with respect to volume for a single dispersed bubble, which for spherical bubbles is,

$$\left(\frac{\mathrm{d}a_{ib}}{\mathrm{d}v_b}\right) = \frac{4}{\bar{D}_b}. \qquad [164]$$

Using for the average bubble diameter, $\bar{D}_b$ in [164], the Sauter mean diameter of the bubbles,

$$\frac{6}{\bar{D}_b} = \frac{A'''_{ib}}{\alpha_b}. \qquad [165]$$

Using eqns [165], [164] and [163], we obtain,

$$\left(\frac{\mathrm{d}A'''_{ib}}{\mathrm{d}t}\right)_C = \tfrac{2}{3}\rho_G A'''_{ib}\left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{u}_b/\rho_G]\right). \qquad [166]$$

Analogously for the bubble clusters,

$$\left(\frac{\mathrm{d}A'''_{iCl}}{\mathrm{d}t}\right)_C = \tfrac{2}{3}\rho_G A'''_{iCl}\left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{u}_{Cl}/\rho_G]\right), \qquad [167]$$

and for the bubble void fraction,

$$\left(\frac{\mathrm{d}\alpha_b}{\mathrm{d}t}\right)_C = \alpha_b \rho_G \left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{u}_b/\rho_G]\right), \qquad [168]$$

thus,

$$\left(\frac{\mathrm{d}\alpha_b}{\mathrm{d}t}\right)_C = \alpha_b \rho_G \left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{u}_b/\rho_G]\right). \qquad [169]$$

The last source term is the vapor generation at a solid surface (Kelly 1993)

$$\left(\frac{dA'''_{ib}}{\mathrm{d}t}\right)_\Gamma = \frac{1}{\rho_G}\left(\frac{a_{in}}{v_n}\right)\Gamma_w, \qquad [170]$$

where $a_n$ and $v_n$ are the interfacial surface area and volume of the bubbles caused by nucleation at the wall, respectively. We may assume that no clusters are produced instantly from vapor generation, thus,

$$\left(\frac{\mathrm{d}A'''_{iCl}}{\mathrm{d}t}\right)_\Gamma = 0 \qquad [171]$$

and

$$\left(\frac{\mathrm{d}\alpha_b}{\mathrm{d}t}\right)_\Gamma = \frac{1}{\rho_G}\Gamma_w. \qquad [172]$$

We summarize the recommended transport equations in table 2. The coefficients, which are used in these equations, are summarized in table 3. The steady-state values, which are the reference in the transport equation, may be found from the equations summarized in table 1.

Table 2. The first order relaxation model

Transport equation for:

*The interfacial area density of the bubbles outside the bubble clusters*

$$\frac{\partial A'''_{ib}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b A'''_{ib}] = a_{11}(A'''_{ib} - A'''_{ib\infty}) + a_{12}\frac{A'''_{ib\infty}}{A'''_{iCl\infty}}(A'''_{iCl} - A'''_{iCl\infty}) + a_{13}\frac{A'''_{ib\infty}}{\alpha_{b\infty}}(\alpha_b - \alpha_{b\infty}) + \left(\frac{dA'''_{ib}}{dt}\right)_C + \left(\frac{dA'''_{ib}}{dt}\right)_\Gamma$$

*The interfacial area density of the bubbles inside the bubble clusters*

$$\frac{\partial A'''_{iCl}}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_{Cl} A'''_{iCl}] = a_{21}\frac{A'''_{iCl\infty}}{A'''_{ib\infty}}(A'''_{ib} - A'''_{ib\infty}) + a_{22}(A'''_{iCl} - A'''_{iCl\infty}) + a_{23}\frac{A'''_{iCl\infty}}{\alpha_{b\infty}}(\alpha_b - \alpha_{b\infty}) + \left(\frac{dA'''_{iCl}}{dt}\right)_C$$

*The void fraction of bubbles outside of clusters*

$$\frac{\partial \alpha_b}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b \alpha_b] = a_{31}\frac{\alpha_{b\infty}}{A'''_{ib\infty}}(A'''_{ib} - A'''_{ib\infty}) + a_{32}\frac{\alpha_{b\infty}}{A'''_{iCl\infty}}(A'''_{iCl} - A'''_{iCl\infty}) + a_{33}(\alpha_b - \alpha_{b\infty}) + \left(\frac{d\alpha_b}{dt}\right)_C + \left(\frac{d\alpha_b}{dt}\right)_\Gamma$$

The source terms:

*Due to compressibility of the gas*

$$\left(\frac{dA'''_{ib}}{dt}\right)_C = \tfrac{2}{3}\rho_G A'''_{ib}\left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b/\rho_G]\right)$$

$$\left(\frac{dA'''_{iCl}}{dt}\right)_C = \tfrac{2}{3}\rho_G A'''_{iCl}\left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_{Cl}/\rho_G]\right)$$

$$\left(\frac{d\alpha_b}{dt}\right)_C = \alpha_b\rho_G\left(\frac{\partial(1/\rho_G)}{\partial t} + \nabla \cdot [\bar{\mathbf{u}}_b/\rho_G]\right)$$

*Due to vapor generation at a solid surface*

$$\left(\frac{dA'''_{ib}}{dt}\right)_\Gamma = \frac{1}{\rho_G}\left(\frac{a_n}{v_n}\right)\Gamma_w$$

$$\left(\frac{d\alpha_b}{dt}\right)_\Gamma = \frac{1}{\rho_G}\Gamma_w$$

Table 3. The coefficients for the transport equations

The coefficients

$$a_{11} = -8r_2\frac{\alpha_b}{\bar{v}_b} - r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}$$

$$a_{12} = r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}\left(-3 + \frac{1}{4\chi(1+\chi)}\left[2(1-\chi)^2 + \frac{r_8}{r_5+r_8}(2+5\chi)\right]\right)$$

$$a_{13} = \tfrac{1}{6}r_1\bar{v}_b + 4r_2\frac{\alpha_b}{\bar{v}_b} - \tfrac{2}{3}r_{13}\frac{\alpha_b}{\bar{v}_b}\frac{1}{\chi(2-\chi)}\left[2(1-\chi)^2 + \frac{r_8}{r_5+r_8}\right] + \tfrac{2}{3}a_{12}\frac{\alpha_b}{\alpha_{Cl}}$$

$$a_{21} = 4r_2\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}\left(1 + 2\frac{\bar{v}_{Cl}}{\bar{v}_b}\frac{(1+\chi)}{(2-\chi)}\right)$$

$$a_{22} = 2r_{13}\frac{\alpha_b}{\bar{v}_{Cl}} - r_{13}\frac{\alpha_b}{\bar{v}_b}\frac{1-\chi}{\chi(1+\chi)}$$

$$a_{23} = -\tfrac{4}{3}r_2\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}\left(1 + 4\frac{\bar{v}_{Cl}}{\bar{v}_b}\frac{(1+\chi)}{(2-\chi)}\right) + \tfrac{2}{3}r_{13}\frac{\alpha_b^2}{\alpha_{Cl}\bar{v}_b}\left[\frac{(1-2\chi)(1-\chi)}{(2-\chi)(1+\chi)} + \frac{r_8/(r_5+r_8)}{\chi(2-\chi)}\right] + \tfrac{2}{3}r_{13}\frac{\alpha_b}{\bar{v}_{Cl}}\left(1 + 2\frac{\alpha_b}{\alpha_{Cl}}\right)$$

$$a_{31} = -6r_2\frac{\alpha_b}{\bar{v}_b}$$

$$a_{32} = -3r_{13}\frac{\alpha_{Cl}}{\bar{v}_{Cl}}$$

$$a_{33} = 2r_2\frac{\alpha_b}{\bar{v}_b} - r_{13}\frac{\alpha_b}{\bar{v}_b}\left(\frac{2(1-\chi)^2 + r_8/(r_5+r_8)}{\chi(2-\chi)}\right) - r_{13}\frac{1+\alpha_b}{\bar{v}_{Cl}}$$

## 6. SUMMARY AND CONCLUSIONS

In order to predict the interfacial area density for bubbly two-phase flows, it is sufficient to consider only individual bubble coalescence and break-up. Indeed, bubble clusters also have to be taken into account. The known models for bubble coalescence and break-up may be extended to bubble clusters resulting in Boltzmann transport equations for both the bubbles and the bubble clusters. The steady-state values describing the size distribution and number of bubbles both inside and outside of the bubble clusters may be evaluated from such an extended model. In addition, we may obtain a first order relaxation model for the interfacial area densities of the bubbles and the bubble clusters if we write the transport equations for small deviations from their steady-state values. These transport equations are very convenient for the numerical evaluation of transient multiphase flows using multidimensional two fluid models.

## REFERENCES


Akita, K. & Yoshida, F. 1974 Bubble size, interfacial area, and liquid-phase mass transfer coefficient in bubble columns. *Ind. Eng. Chem. Process Des. Develop.* **13**, 84–91.

Bensler, H., Delhaye, J. M. & Favreau, C. 1991 Determination of the volumetric interfacial area, volumetric void fraction and Sauter mean diameter in bubbly flow by means of ultrasound attenuation. *Proc. World Conf. on Experimental Heat Transfer, Fluid Mechanics, and Thermodynamics*, pp. 1096–1104.

Burgess, J. & Calderbank, P. H. 1975 The measurement of bubble parameters in two-phase dispersions—I. *Chem. Eng. Sci.* **30**, 743–750.

Burgess, J. & Calderbank, P. H. 1975 The measurement of bubble parameters in two-phase dispersions—II. *Chem. Eng. Sci.* **30**, 1107–1121.

Burgess, J. & Calderbank, P. H. 1975 The measurement of bubble parameters in two-phase dispersions—III. *Chem. Eng. Sci.* **30**, 1511–1518.

Calderbank, P. 1967 Gas absorption from bubbles. *The Chem. Eng.* **43**, CE209–CE233.

Chandrasekharan, K. & Calderbank, P. H. 1981 Further observations on the scale-up aerated mixing vessels. *Chem. Eng. Sci.* **36**, 819–823.

Chang, J. & Harvel, G. D. 1992 Determination of gas–liquid bubble column instantaneous interfacial area and void fraction by a real-time neutron radiography method. *Chem. Eng. Sci.* **47**, 13–14; 3639–3646.

Daly, J., Patel, S. A. & Bukur, D. B. 1992 Measurement of gas holdups and sauter mean bubble diameters in bubble column reactors by dynamic gas disengagement method. *Chem. Eng. Sci* **47**, 13–14; 3647–3654.

Deckwer, W.-D. 1992 *Bubble Columns*. Otto Salle Verlag, Frankfurt, Germany.

de Figueiredo, M. L. & Calderbank, P. H. 1979 The scale-up of aerated mixing vessels for specified oxygen dissolution rates. *Chem. Eng. Sci.* **34**, 1333–1338.

Garncarek, Z., Idzik, J. & Spisak, W. 1991 A new method in investigations of bubble cluster shapes in two-phase flow. *Chem. Eng. Process.* **30**, 107–112.

Guido-Lavalle, G., Clausse, A. & Converti, J. 1993 A contribution to the kinetic formulation of gas-liquid flows. *Chem. Eng. Communications* **120**, 99.

Guido-Lavalle, G., Garcia, P., Clausse, A. & Qazi, M. K. 1994 A bubble number density constitutive equation. *Nuclear Eng. Design* **152**, 213–224.

Ishii, M. 1975 *Thermo-fluid Dynamic Theory of Two-phase Flow*. Eyrolles, Paris.

Ishii, M. & Mishima, K. 1984 Two-fluid model and hydrodynamic constitutive relations. *Nucl. Eng. Des.* **82**, 107–126.

Ishii, M. 1992 Interfacial area in two-phase flow. HTD Vol. 197, pp. 1–5. *Two-phase Flow and Heat Transfer*, ASME.

Kalkach-Navarro, S. 1992 Two-fluid modeling of the bubbly/slug flow regime transition. Ph.D. Thesis, Rensselear Polytechnic Institute, Troy.

Kataoka, I., Ishii, M. & Serizawa, A. 1986 Local formulation and measurements of interfacial area concentration in two-phase flow. *Int. J. Mult. Flow* **12**, 505–529.

Kelly, J. 1993 A four-field approach towards developing a mechanistic model for multiphase flow. Pacific Northwest Laboratory Report PNL-SA-18878, DE93 006572.

Lahey, R. T. & Lee, S. J. 1992 Phase distribution and two-phase turbulence for bubbly flows in pipe. In *Multiphase Science and Technology*, Vol. 6, pp. 303–349. Hemisphere Publishing Corp., Washington.

Levich, V. 1962 *Physiochemical Hydrodynamics*. Prentice Hall, Englewood Cliffs, NJ.

Lewis, D. A. & Davidson, J. F. 1983 Bubble sizes produced by shear and turbulence in a bubble column. *Chem. Eng. Sci.* **38,** 161–167.

Marrucci, G. & Nicodemo, L. 1967 Coalescence of gas bubbles in aqueous solutions of inorganic electrolytes. *Chem. Eng. Sci.* **22,** 1257–1265.

Mudde, R., Bakker, R. A. & van den Akker, H. E. A. 1992 Noise analysis of transmitted light beams for determining bubble velocity and gas holdup profiles in a bubble column. *Chem. Eng. Sci.* **47,** 13–14; 3631–3638.

Nagel, O., Kürten, H. & Sinn, R. 1972 Stoffaustauschfläche und Energiedissipationsdichte als Auswahlkriterien für Gas/Flüssig-Reaktoren: Teil 1. *Chem. Ing. Tech.* **44,** 367–373.

Nagel, O., Kürten, H. & Sinn, R. 1972 Stoffaustauschfläche und Energiedissipationsdichte als Auswahlkriterien für Gas/Flüssigkeits-Reaktoren: Teil 2. *Chem. Ing. Tech.* **44,** 899–903.

Nagel, O., Kürten, H. & Hegner, B. 1973 Die Stoffaustauschfläche in Gas/Flüssigkeits-Kontaktapparaten Auswahlkriterien und Unterlagen zur Vergrößerung. *Chem. Ing. Tech.* **45,** 913–920.

Nagel, O. & Kürten, H. 1976 Untersuchungen zum Dispergieren im turbulenten Scherfeld. *Chem. Ing. Tech.* **48,** 513–519.

Nagel, O., Hegner, B. & Kürten, H. 1978 Kriterien für die Auswahl und die Auslegung von Gas/Flüssigkeits-Reaktoren. *Chem. Ing. Tech.* **50,** 935–944.

Nagel, O., Kürten, H. & Hegner, B. 1978 Design of gas/liquid reactors: mass transfer area and input of energy. *Proc. ICHMT*, pp. 835–876. Hemisphere Publishing Corp., Washington.

Oolman, T. & Blanch, H. W. 1986 Bubble coalescence in air-sparged bioreactors. *Chemical Engng J.* **32,** B7–B17.

Park, J. 1992 Void wave propagation in two-phase flow. Ph.D. Thesis, RPI, Troy.

Patel, S., Daly, J. G. & Bukur, D. B. 1989 holdup and interfacial area measurements using dynamic gas disengagements. *AIChE J.* **35,** 931–942.

Prince, M. & Blanch, H. W. 1990 Transition electrolyte concentrations for bubble coalescence. *AIChE J.* **36,** 1425–1429.

Prince, M. & Blanch, H. W. 1990b Bubble coalescence and break-up in air sparged bubble columns. *AIChE J.* **36,** 1485–1499.

Schumpe, A. & Grund, G. 1986 The gas desengagement technique for studying gas holdup structure in bubble columns. *Can J. Chem. Eng.* **64,** 891–896.

Serizawa, A. & Kataoka, I. 1990 Phase distribution in two-phase flow. *Int. Conf. Heat Mass Transfer*, Dubrovnik.

Stewart, C., Crowe, C. T. & Saunders, S. C. 1993 A model for simultaneous coalescence of bubble clusters. *Chem. Eng. Sci.* **48,** 3347–3354.

Thomas, N., Auton, T. R., Sene, K. & Hunt, J. C. R. 1983 Physical modelling of multi-phase flow. *BHRA Fluid Engng.* 169–184.